

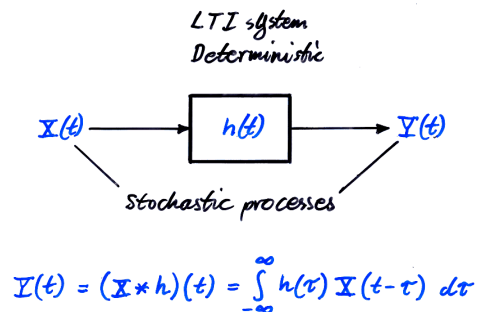
TSDT14 Signal Theory

Lecture 3

LTI Filtering, White Noise, Colored Noise

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Filtering Stochastic Processes



Demand: stability $\because \int_{-\infty}^{\infty} |h(t)| dt$ convergent.

Holds regardless of stationarity.

Expectation of the Output (Input WSS)

Notation: $H(f) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt$

Expectation: $m_Y(t) = E\{Y(t)\} = E\left\{\int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau\right\}$

Expectation is linear

$$= \int_{-\infty}^{\infty} h(\tau) \cdot E\{X(t-\tau)\} d\tau = \int_{-\infty}^{\infty} h(\tau) \cdot m_X(t-\tau) d\tau$$

X(t) WSS *Identify*

$$= m_X \cdot \int_{-\infty}^{\infty} h(\tau) d\tau = m_X \cdot H(0)$$

Thus: $m_Y(t)$ is independent of t .

ACF of the Output (Input WSS)

ACF: $r_Y(t, t+\tau) = E\{Y(t)Y(t+\tau)\} = E\left\{\int_{-\infty}^{\infty} h(\tau_1) X(t-\tau_1) d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2) X(t+\tau-\tau_2) d\tau_2\right\}$

E{...} linear

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) E\{X(t-\tau_1) X(t+\tau-\tau_2)\} d\tau_1 d\tau_2$$

X(t) WSS

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) r_X(\tau_2 - \tau_1 + \tau) d\tau_1 d\tau_2 = \int_{-\infty}^{\infty} \tilde{h}(\tau_3) h(\tau_2) r_X(\tau - \tau_3 - \tau_2) d\tau_2 d\tau_3 = (h * \tilde{h} * r_X)(\tau)$$

$\tilde{h}(t) = h(-t)$
 $\tau_3 = -\tau_1$

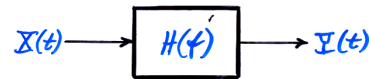
Thus: $r_Y(t, t+\tau)$ independent of t , and we write

$$r_Y(\tau) = (h * \tilde{h} * r_X)(\tau)$$

PSD: $R_Y(f) = H(f) \cdot H^*(f) \cdot R_X(f) = |H(f)|^2 R_X(f)$

Example Filtering

Let $X(t)$ be a wide sense stationary process with $r_X(\tau) = e^{-|\tau|}$



$$H(f) = \begin{cases} 1, & |f| \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Determine the output power $E\{Y^2(t)\}$.

$$\begin{aligned} E\{Y^2(t)\} &= r_Y(0) = \int_{-\infty}^{\infty} R_Y(f) df = \int_{-\infty}^{\infty} |H(f)|^2 \cdot R_X(f) df \\ &= \int_{-1}^1 R_X(f) df \quad \text{Table} \\ &= \int_{-1}^1 \frac{2}{1+(2\pi f)^2} df = \int_{-2\pi}^{2\pi} \frac{1}{1+\omega^2} d\omega \quad \text{Std integral} \\ &= \frac{1}{\pi} [\arctan(\omega)]_{-2\pi}^{2\pi} = \frac{2}{\pi} \arctan(2\pi) \end{aligned}$$

Normalized Filters

Normalized filter: $\int_{-\infty}^{\infty} |h(t)|^2 dt = 1$
Input: $X(t)$, WGN, $R_X(f) = R_0$, $m_X = 0$.

Output:

$$R_Y(f) = |H(f)|^2 R_X(f)$$

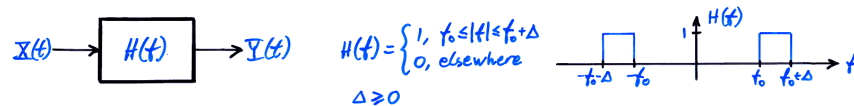
$$m_Y = H(0) \cdot m_X = 0$$

$$\begin{aligned} \sigma_Y^2 &= E\{Y^2(t)\} = r_Y(0) = \int_{-\infty}^{\infty} R_Y(f) df \\ &= R_0 \cdot \int_{-\infty}^{\infty} |H(f)|^2 df = R_0 \cdot \int_{-\infty}^{\infty} |h(t)|^2 dt = R_0 \end{aligned}$$

Parseval Normalized filter

And LTI + Gaussian input $\Rightarrow Y(t)$ Gaussian.

Argument for $R_X(f) \geq 0$



$$\begin{aligned} 0 \leq E\{Y^2(t)\} &= \int_{-\infty}^{\infty} |H(f)|^2 R_X(f) df = 2 \cdot \int_{f_0}^{f_0+\Delta} R_X(f) df \\ &\approx 2 \cdot \Delta \cdot R_X(f_0) \quad \Delta \text{ small} \\ \Rightarrow R_X(f_0) &\approx \frac{E\{Y^2(t)\}}{2 \cdot \Delta} \geq 0 \end{aligned}$$

More precisely:

$$R_X(f_0) = \lim_{\Delta \rightarrow 0} \frac{E\{Y^2(t)\}}{2 \Delta} \geq 0$$

Cross-Correlation

Two Processes: $X(t)$ & $Y(t)$

Cross-Correlation: $r_{X,Y}(t_1, t_2) = E\{X(t_1)Y(t_2)\}$

Properties:

$$r_{X,X}(t_1, t_2) = r_X(t_1, t_2)$$

$$r_{X,Y}(t_1, t_2) = r_{Y,X}(t_2, t_1)$$

$$r_{X,Y}^2(t_1, t_2) \leq r_X(t_1, t_1) \cdot r_Y(t_2, t_2)$$

Uncorrelated and Independent Processes

Definition: Consider two processes $X(t)$ and $Y(t)$ and sample them in the time instances $\bar{t}_1 = (t_{11}, \dots, t_{1N})$ and $\bar{t}_2 = (t_{21}, \dots, t_{2N})$, respectively. The processes are said to be independent if $X(\bar{t}_1)$ and $Y(\bar{t}_2)$ are independent, i.e. if

$$F_{X(\bar{t}_1), Y(\bar{t}_2)}(\bar{x}, \bar{y}) = F_{X(\bar{t}_1)}(\bar{x}) \cdot F_{Y(\bar{t}_2)}(\bar{y})$$

holds for every N , every $\bar{t}_1$ and every $\bar{t}_2$.

Definition: Two processes $X(t)$ and $Y(t)$ are said to be uncorrelated if

$$r_{XY}(t_1, t_2) = m_X(t_1) \cdot m_Y(t_2)$$

holds for all t_1 and t_2 .

Relation: Independent $\Rightarrow$ Uncorrelated.

Joint Stationarity

Definition: The processes $X(t)$ and $Y(t)$ are said to be jointly stationary in the wide sense if

$X(t)$ is stationary in the wide sense,
 $Y(t)$ is stationary in the wide sense,
 $r_{XY}(t_1, t_2)$ depends only on $t_1 - t_2$.

Notation: $r_{X,Y}(t_1 - t_2) = r_{X,Y}(t_1, t_2)$

$$r_{X,Y}(\tau) = r_{X,Y}(t + \tau, t)$$

Cross spectrum: $R_{X,Y}(f) = \mathcal{F}\{r_{X,Y}(\tau)\}$

Jointly Gaussian Processes

Definition: Consider two processes $X(t)$ and $Y(t)$ and sample them in the time instances $\bar{t}_1 = (t_{11}, \dots, t_{1N})$ and $\bar{t}_2 = (t_{21}, \dots, t_{2N})$, respectively. The processes are said to be jointly Gaussian if $[X(\bar{t}_1), Y(\bar{t}_2)]$ are jointly Gaussian for every N , every $\bar{t}_1$ and every $\bar{t}_2$.

Theorem: If $X(t)$ and $Y(t)$ are uncorrelated and jointly Gaussian, they are also independent.

LTI Filtering

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t) = (X * h)(t) = \int_{-\infty}^{\infty} h(\tau) X(t - \tau) d\tau$$

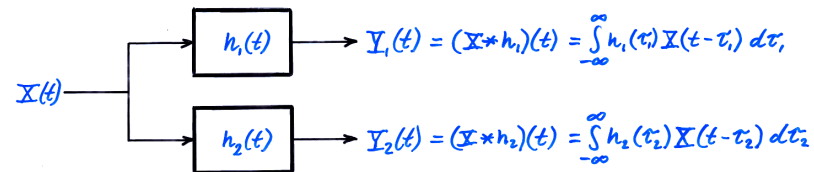
$$\begin{aligned} r_{Y,X}(t_1, t_2) &= E\{Y(t_1)X(t_2)\} = E\left\{\int_{-\infty}^{\infty} h(\tau) X(t_1 - \tau) d\tau \cdot X(t_2)\right\} \\ &= \int_{-\infty}^{\infty} h(\tau) E\{X(t_1 - \tau)X(t_2)\} d\tau = \int_{-\infty}^{\infty} h(\tau) r_X(t_1 - \tau, t_2) d\tau \\ &\stackrel{\text{w.s. stat.}}{=} \int_{-\infty}^{\infty} h(\tau) r_X(t_1 - t_2 - \tau) d\tau = (h * r_X)(t_1 - t_2) \end{aligned}$$

Jointly stationary in the wide sense?

$$r_{Y,X}(\tau) = (h * r_X)(\tau)$$

$$R_{Y,X}(f) = H(f) \cdot R_X(f)$$

Filtering with Orthogonal Filters 1(2)



Orthogonal: $\int_{-\infty}^{\infty} h_1(t) h_2(t) dt = 0$

Input: $X(t)$, WGN, $R_X(t) = R_0$, $m_X = 0$.

Means: $m_{Y_k} = H_k(0) m_X = 0$

Question: Are they correlated? (same time).

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Filtering with Orthogonal Filters 2(2)

$$\begin{aligned}
 r_{Y_1, Y_2}(t, t) &= E\{Y_1(t) Y_2(t)\} = E\left\{\int_{-\infty}^{\infty} h_1(\tau_1) X(t - \tau_1) d\tau_1 \cdot \int_{-\infty}^{\infty} h_2(\tau_2) X(t - \tau_2) d\tau_2\right\} \\
 &= \int_{-\infty}^{\infty} h_1(\tau_1) \cdot \int_{-\infty}^{\infty} h_2(\tau_2) \cdot E\{X(t - \tau_1) X(t - \tau_2)\} d\tau_2 d\tau_1 \\
 &= \int_{-\infty}^{\infty} h_1(\tau_1) \cdot \int_{-\infty}^{\infty} h_2(\tau_2) \cdot r_X(\tau_1 - \tau_2) d\tau_2 d\tau_1 \\
 &\stackrel{\text{white}}{\downarrow} = \int_{-\infty}^{\infty} h_1(\tau_1) \cdot \int_{-\infty}^{\infty} h_2(\tau_2) \cdot R_0 \delta(\tau_1 - \tau_2) d\tau_2 d\tau_1 = R_0 \cdot \int_{-\infty}^{\infty} h_1(\tau_1) \cdot (h_2 * \delta)(\tau_1) d\tau_1 \\
 &= R_0 \cdot \int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_1) d\tau_1 \stackrel{\text{orthogonal}}{\downarrow} = 0 = m_{Y_1} \cdot m_{Y_2}
 \end{aligned}$$

Uncorrelated + jointly Gaussian $\Rightarrow$ Independent.