

TSDT14 Signal Theory

Lecture 1

Introduction and Repetition

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TSDT14 Signal Theory - Formalia

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Examination: Laborations (2hp):
Study 1, 2, 3, 4 (4x2 hours)
Sign-up on the web
Report before end of exam period

Written exam (4hp):
1 simple task – Demand: 2/3 OK
5 tasks (5 points each), max 25
Pass: 10 points

Course Aims 1(2)

After passing the course, the student should

- be able to clearly define central concepts regarding stochastic processes, using own words. **(task 1)**
- be able to reliably perform standard calculations regarding stochastic processes, e.g. LTI filtering (both time continuous and time discrete), sampling and pulse amplitude modulation. **(task 1)**
- be able to reliably perform standard calculations regarding stochastic processes being exposed to certain momentary non-linearities that are common in telecommunication, especially uniform quantization and monomial non-linearities of low degrees. **(task 1)**
- with some reliability be able to solve problems that demand integration of knowledge from different parts of the course. **(tasks 2-6)**

Course Aims 2(2)

After passing the course, the student should

- be able to account for the connection between different concepts in the course in a structured way using adequate terminology. **(lab report)**
- be able to estimate the auto correlation function and power spectral density of a stochastic process based on a realization of the process. Also, clearly and logically account for those estimations and conclusions that can be drawn from them. **(lab report)**

Tutorial Sessions (Teaching Sessions)

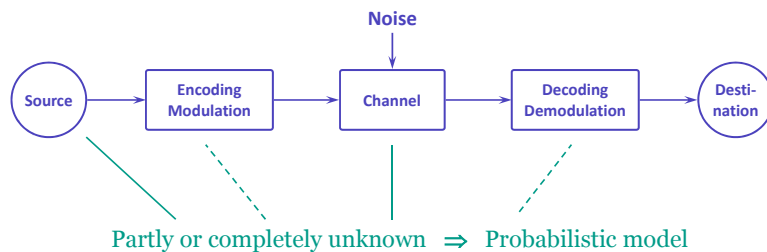
All tutorial are given in English.

- Group A: Zheng Chen
- Group B: Amin Ghazanfari
- Group C: Özgecan Özdoğan

You are free to follow any tutorial group you wish. Please try to spread out over the groups.

Individual teaching can be done in any language that works.

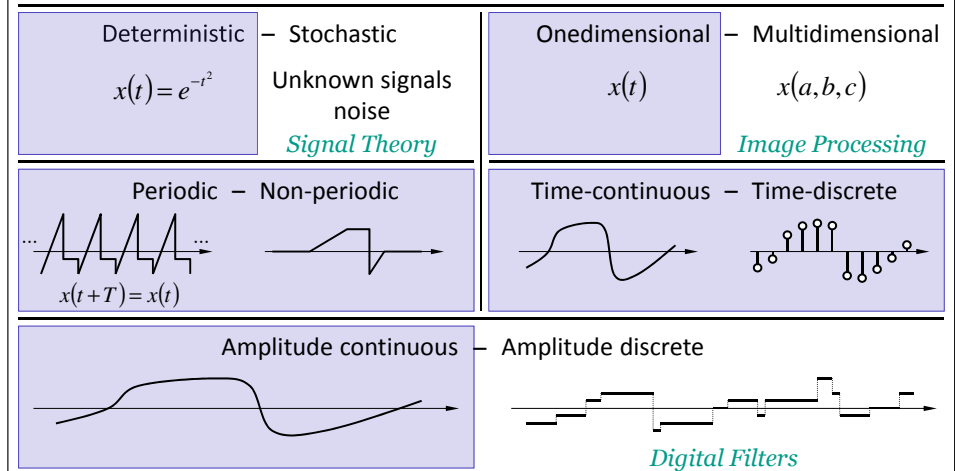
Standard Situation



We can have

Linear and non-linear filtering
Sampling and reconstruction
Up- and down-sampling
Modulation
Also: Error Correction, Packing, Cryptology,...

Classification of Signals



Classification of Systems

Linearity – Preserves linear combinations.

Time-invariance – Behaves the same way all the time.

Causality – Has no knowledge about the future.

Stability – Bounded input results in bounded output.

LTI – Linear and Time-Invariant. Convolution.

Frequency Domain



Jean Baptiste Joseph Fourier
1768 – 1830

Time-Continuous Fourier transform:

Signal: $x(t)$

Spectrum: $X(f) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

Inverse: $x(t) = F^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$

Time-Discrete Fourier transform:

Signal: $x[n]$

Spectrum: $X[\theta] = F\{x[n]\} = \sum_n x[n] e^{-j2\pi \theta n}$

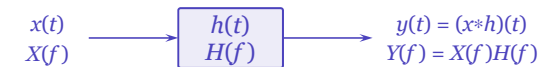
Inverse: $x[n] = F^{-1}\{X[\theta]\} = \int_0^1 X[\theta] e^{j2\pi \theta n} d\theta$

Output from an LTI System

Notation: $A(f) = \mathcal{F}\{a(t)\}$ $B(f) = \mathcal{F}\{b(t)\}$

Property: $\mathcal{F}\{(a * b)(t)\} = \int_{-\infty}^{\infty} (a * b)(t) e^{-j2\pi f t} dt = A(f)B(f)$

LTI System:



Signal Power and Signal Energy – Parseval

Signal power: $|x(t)|^2$ Signal energy: $\int_{-\infty}^{\infty} |x(t)|^2 dt$

Parseval's relation (special case):

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) x^*(t) dt = \int_{-\infty}^{\infty} X(f) X^*(f) df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Energy spectrum: $|X(f)|^2$

Parseval's relation (generally): $\int_{-\infty}^{\infty} a(t) b^*(t) dt = \int_{-\infty}^{\infty} A(f) B^*(f) df$

The Principle sine in – sine out

For LTI systems, we have:

Sine in – Sine out (the same frequency)

More precisely:

Input: $x(t) = \hat{X} \sin(2\pi f_0 t + \varphi)$

Output: $y(t) = \hat{X} |H(f_0)| \sin(2\pi f_0 t + \varphi + \arg\{H(f_0)\})$

Amplitude characteristic: $|H(f)|$

Phase characteristic: $\arg\{H(f)\}$

This is the $j\omega$ method in condensed form.

Classification of Frequency Selective Filters

- A (frequency selective) filter is an LTI system. Usually it lets some frequency band through or stops it.
- Usually an electrical network – either passive or active.

Notation	Ideal amplitude characteristics	Real amplitude char.
Lowpass filter (LP filter):		
Highpass filter (HP filter):		
Bandpass filter (BP filter):		
Bandstop filter (BS filter):		
Allpass filter (AP filter):		

Example

Game based on tossing two fair coins:

2 heads	+400
2 tails	-100
1 tail, 1 head	-200

$$\Omega = \{HH, HT, TH, TT\}$$

$$\Omega_X = \{-200, -100, 400\}$$

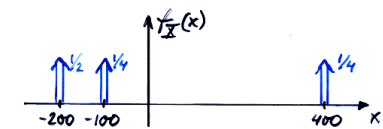
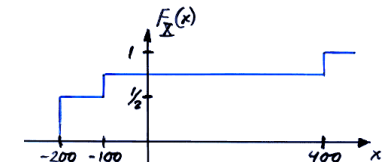
$$X(\omega) = \begin{cases} 400, & \omega = HH \\ -100, & \omega = TT \\ -200, & \omega \in \{HT, TH\} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < -200 \\ 1/2, & -200 \leq x < -100 \\ 3/4, & -100 \leq x < 400 \\ 1, & x \geq 400 \end{cases}$$

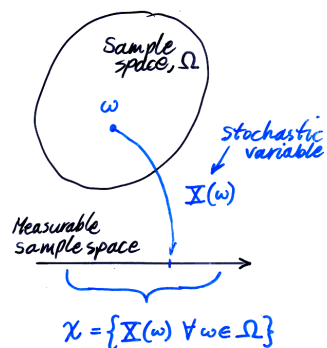
$$= \frac{1}{2}u(x+200) + \frac{1}{4}u(x+100) + \frac{1}{4}u(x-400)$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \frac{1}{2}\delta(x+200) + \frac{1}{4}\delta(x+100) + \frac{1}{4}\delta(x-400)$$



Probabilities and Distributions



Probability: $\Pr\{A\} \in [0, 1]$

Joint prob.: $\Pr\{A, B\}$

Cond. Prob.: $\Pr\{A | B\} = \frac{\Pr\{A, B\}}{\Pr\{B\}}$

Prob. distr.: $F_X(x) = \Pr\{X \leq x\} \in [0, 1]$

Prob. density.: $f_X(x) = \frac{d}{dx} F_X(x)$

Properties: $F_X(x)$ is non-decreasing

$$f_X(x) \geq 0 \text{ for all } x$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

$$\Pr\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x) dx$$

Expectations

Expectation (mean):

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx$$

For discrete variables also:

$$E\{X\} = \sum_i x_i \cdot \Pr\{X = x_i\}$$

For functions of a variable:

$$Y = g(X)$$

$$E\{Y\} = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Also:

$$m_X, m_Y$$

Quadratic mean (Power):

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

Variance:

$$\text{Var}\{X\} = E\{(X - m_X)^2\} = E\{X^2\} - m_X^2$$

Also:

$$\sigma_X^2$$

Standard deviation:

$$\sigma_X$$

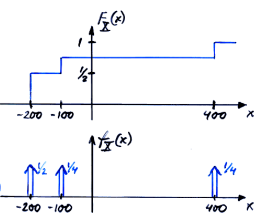
Example cont'd

$$F_X(x) = \begin{cases} 0, & x < -200 \\ 1/2, & -200 \leq x < -100 \\ 3/4, & -100 \leq x < 400 \\ 1, & x \geq 400 \end{cases}$$

$$= \frac{1}{2} u(x+200) + \frac{1}{4} u(x+100) + \frac{1}{4} u(x-400)$$

$$f_Y(x) = \frac{d}{dx} F_Y(x)$$

$$= \frac{1}{2} \delta(x+200) + \frac{1}{4} \delta(x+100) + \frac{1}{4} \delta(x-400)$$



$$Y = \text{sgn} X$$

$$\Pr\{Y=1\} = \Pr\{X>0\} = \Pr\{X=400\} = 1/4$$

$$\Pr\{Y = -1\} = \Pr\{X < 0\} = \Pr\{X = -100\} + \Pr\{X = -200\} = 3/4$$

$$F_{\pm}(y) = \frac{3}{4}u(y+1) + \frac{1}{4}u(y-1)$$

$$f_{\psi}(y) = \frac{3}{4} \delta(y+1) + \frac{1}{4} \delta(y-1)$$

$$E\{X\} = \int_{-\infty}^{\infty} x \left(\frac{1}{2} \delta(x+200) + \frac{1}{4} \delta(x+100) + \frac{1}{4} \delta(x-400) \right) dx$$

$$= -200 \cdot \frac{1}{2} - 100 \cdot \frac{1}{4} + 400 \cdot \frac{1}{4} = -25$$

$$E\{X^2\} = (-200)^2 \cdot \frac{1}{2} + (-100)^2 \cdot \frac{1}{4} + (400)^2 \cdot \frac{1}{4} = 62500$$

$$\text{Var}\{X\} = E\{X^2\} - (E\{X\})^2 = 61875$$

Example of the Q Function

x	0	1	2	3	4	5	6	7	8	9	exp
0.0	5.0000	4.9601	4.9202	4.8803	4.8405	4.8006	4.7608	4.7210	4.6812	4.6414	
0.1	4.6017	4.5620	4.5224	4.4828	4.4433	4.4038	4.3644	4.3251	4.2858	4.2465	
0.2	4.2074	4.1683	4.1294	4.0905	4.0517	4.0129	3.9743	3.9358	3.8974	3.8591	
0.3	3.8209	3.7828	3.7448	3.7070	3.6693	3.6317	3.5942	3.5569	3.5197	3.4827	
0.4	3.4458	3.4090	3.3724	3.3360	3.2997	3.2636	3.2276	3.1918	3.1561	3.1207	
0.5	3.0854	3.0503	3.0153	2.9806	2.9460	2.9116	2.8774	2.8434	2.8096	2.7760	
0.6	2.7425	2.7093	2.6763	2.6435	2.6109	2.5785	2.5463	2.5143	2.4825	2.4510	-1
0.7	2.4196	2.3885	2.3576	2.3270	2.2965	2.2663	2.2363	2.2065	2.1770	2.1476	
0.8	2.1186	2.0897	2.0611	2.0327	2.0045	1.9766	1.9489	1.9215	1.8943	1.8673	
0.9	1.8406	1.8141	1.7879	1.7617	1.7361	1.7106	1.6853	1.6602	1.6354	1.6109	
1.0	1.5866	1.5625	1.5386	1.5151	1.4917	1.4686	1.4457	1.4231	1.4007	1.3786	
1.1	1.3567	1.3350	1.3136	1.2924	1.2714	1.2507	1.2302	1.2100	1.1900	1.1702	
1.2	1.1507	1.1314	1.1123	1.0935	1.0749	1.0565	1.0383	1.0204	1.0027	9.8525	
1.3	9.6800	9.5098	9.3418	9.1759	9.0123	8.8508	8.6915	8.5343	8.3793	8.2264	
1.4	8.0757	7.9270	7.7804	7.6359	7.4934	7.3529	7.2145	7.0781	6.9437	6.8112	
1.5	6.6807	6.5522	6.4255	6.3008	6.1780	6.0571	5.9380	5.8208	5.7053	5.5917	
1.6	5.4799	5.3699	5.2616	5.1551	5.0503	4.9471	4.8457	4.7460	4.6479	4.5514	
1.7	4.4565	4.3633	4.2716	4.1815	4.0930	4.0059	3.9204	3.8364	3.7538	3.6727	
1.8	3.5930	3.5148	3.4380	3.3625	3.2884	3.2157	3.1443	3.0742	3.0054	2.9379	-5
1.9	2.8717	2.8067	2.7429	2.6803	2.6190	2.5588	2.4998	2.4419	2.3852	2.3295	
2.0	2.2750	2.2216	2.1692	2.1178	2.0675	2.0182	1.9699	1.9226	1.8763	1.8309	
2.1	1.7864	1.7429	1.7003	1.6586	1.6177	1.5788	1.5386	1.5003	1.4629	1.4262	
2.2	1.3903	1.3553	1.3209	1.2874	1.2545	1.2224	1.1911	1.1604	1.1304	1.1011	
2.3	1.0724	1.0444	1.0170	9.9031	9.6419	9.3867	9.1375	8.8940	8.6563	8.4242	

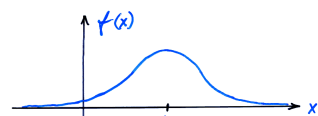
$$Q(1.96) \approx 2.4998 \cdot 10^{-2}$$

Gaussian Distributions, $N(m, \sigma)$

Probability density:

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Probability distribution is hard.



Mean: m

Variance: σ^2

The Q function:

$$Q(x) = 1 - F(x) \text{ for } N(0, 1)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

From a table

[illegible]

Example of the Q Function cont'd

x	0	1	2	3	4	5	6	7	8	9	exp
4.2	1.3346	1.2769	1.2215	1.1685	1.1176	1.0689	1.0221	9.7736	9.3447	8.9337	
4.3	8.5399	8.1627	7.8015	7.4555	7.1241	6.8069	6.5031	6.2123	5.9340	5.6675	
4.4	5.4125	5.1685	4.9350	4.7117	4.4979	4.2935	4.0980	3.9110	3.7322	3.5612	-6
4.5	3.3977	3.2414	3.0920	2.9492	2.8127	2.6823	2.5577	2.4386	2.3249	2.2162	
4.6	2.1125	2.0133	1.9187	1.8283	1.7420	1.6597	1.5810	1.5060	1.4344	1.3660	
4.7	1.3008	1.2306	1.1702	1.1206	1.0656	1.0171	9.6796	9.2113	8.7648	8.3391	
4.8	7.9333	7.5465	7.1779	6.8267	6.4920	6.1731	5.8693	5.5799	5.3043	5.0418	
4.9	4.7918	4.5548	4.3272	4.1115	3.9061	3.7107	3.5247	3.3476	3.1792	3.0190	-7
5.0	2.8665	2.7215	2.5836	2.4524	2.3277	2.2091	2.0963	1.9891	1.8872	1.7903	
5.1	1.6983	1.6108	1.5277	1.4487	1.3737	1.3024	1.2347	1.1705	1.1094	1.0515	

$$Q(x) = 10^{-6} \quad \Rightarrow \quad x \approx 4.75$$

For $x > 0$, we have $(1 - x^{-2})\frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt < Q(x) < \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt$.

For large x we have $Q(x) \approx \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt$.

Other Common Distributions

Uniform distribution:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & \text{elsewhere} \end{cases} \quad \left| \quad \begin{array}{l} \text{Mean: } \frac{a+b}{2} \\ \text{Variance: } \frac{(b-a)^2}{12} \end{array} \right.$$

Exponential distribution:

$$f(x) = \frac{1}{c} e^{-x/c} \cdot u(x) \quad \left| \quad \begin{array}{l} \text{Mean: } c \\ \text{Variance: } c^2 \end{array} \right.$$

Binary distribution:

$$f(x) = p \cdot \delta(x-a) + (1-p) \cdot \delta(x-b) \quad \left| \quad \begin{array}{l} \text{Mean: } p \cdot a + (1-p) \cdot b \\ \text{Variance: } p(1-p) \cdot (b-a)^2 \end{array} \right.$$

Dependencies

Definition: X & Y are independent if $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ holds.

Theorem: Independent $\Leftrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y)$ holds.

Definition: Covariance: $\text{Cov}\{X, Y\} = E\{(X - m_X)(Y - m_Y)\}$

Theorem: $\text{Cov}\{X, Y\} = E\{XY\} - m_X m_Y$

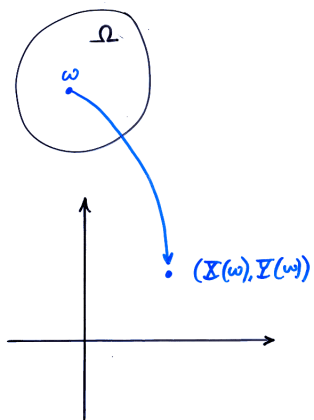
Definition: X & Y are uncorrelated if $\text{Cov}\{X, Y\} = 0$ holds.

Theorem: Independent $\Rightarrow$ uncorrelated.

Note: $\text{Var}\{X\} = \text{Cov}\{X, X\}$

Theorem: Uncorrelated $\Leftrightarrow E\{XY\} = E\{X\}E\{Y\}$
 $\Leftrightarrow \text{Var}\{X+Y\} = \text{Var}\{X\} + \text{Var}\{Y\}$

Two-Dimensional Stochastic Variables



Distribution: $F_{X,Y}(x,y) = \Pr\{X \leq x, Y \leq y\}$

Density: $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$

Properties: $f_{X,Y}(x,y) \geq 0 \quad \forall x, y$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$$

Marginal: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

Function: $E\{g(X, Y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$

Bayes' Rule

Conditional distribution: $F_{X|Y}(x|y) = \frac{\int_{-\infty}^x f_{X,Y}(z,y) dz}{f_Y(y)}$

Conditional density: $f_{X|Y}(x|y) = \frac{d}{dx} F_{X,Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

Bayes rule (continuous): $f_{X|Y}(x|y) = \frac{f_{X,Y}(y|x)}{f_Y(y)} \cdot f_X(x)$

(discrete): $\Pr\{X=x|Y=y\} = \frac{\Pr\{Y=y|X=x\}}{\Pr\{Y=y\}} \cdot \Pr\{X=x\}$

(Mixed): $\Pr\{X=x|Y=y\} = \frac{f_{X,Y}(y|x)}{f_Y(y)} \cdot \Pr\{X=x\}$

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