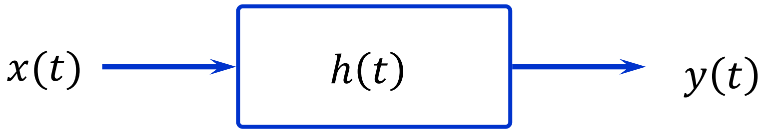
TSKS01 DIGITAL COMMUNICATION

Repetition and Examples

FOURIER TRANSFORM OF SIGNALS AND SYSTEMS

LTI Systems

Definition: A system that is linear and time-invariant is referred to as a *linear time-invariant (LTI) system*.



Definition: The convolution of the signals $a(t)$ and $b(t)$ is denoted by $(a * b)(t)$ and is defined as

$$(a * b)(t) = \int_{-\infty}^{\infty} a(\tau)b(t-\tau)d\tau\,.$$

The convolution is a commutative operation: $(a * b)(t) = (b * a)(t)$.

Output of an LTI Systems

Theorem: Let $x(t)$ be the input to an energy-free LTI system with impulse response $h(t)$, then the output of the system is

$$y(t) = (x * h)(t).$$

Proof: Let $H\{a(t)\}$ denote the output for an arbitrary input $a(t)$, then

$$y(t) = H\{x(t)\} = H\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\}$$

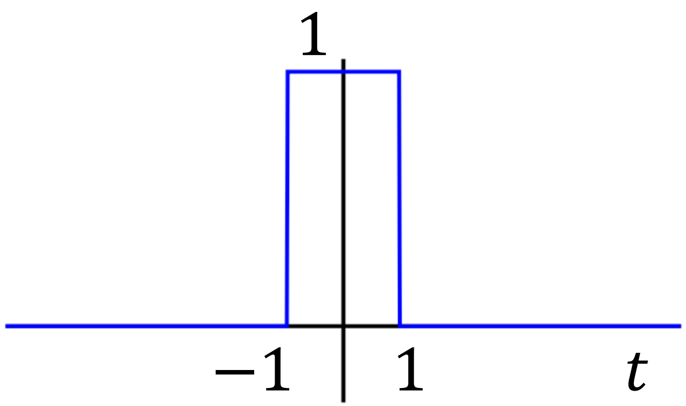
$$\overset{\text{Linear}}{=} \int_{-\infty}^{\infty} x(\tau)\,H\{\delta(t-\tau)\}\,d\tau$$

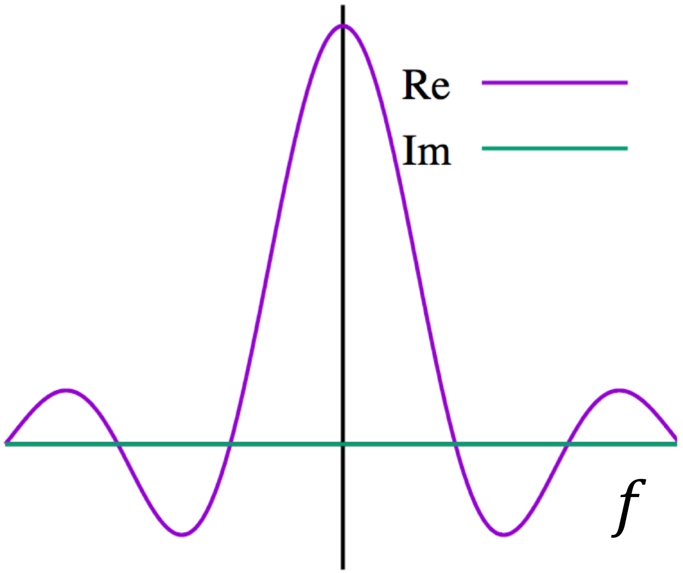
$$\overset{\text{Time-inv}}{=} \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)d\tau = (x * h)(t)$$

Frequency Domain

$$x(t) = u(t+1) - u(t-1)$$

$$X(f) = \mathrm{sinc}(f) = \frac{\sin(\pi f)}{\pi f}$$



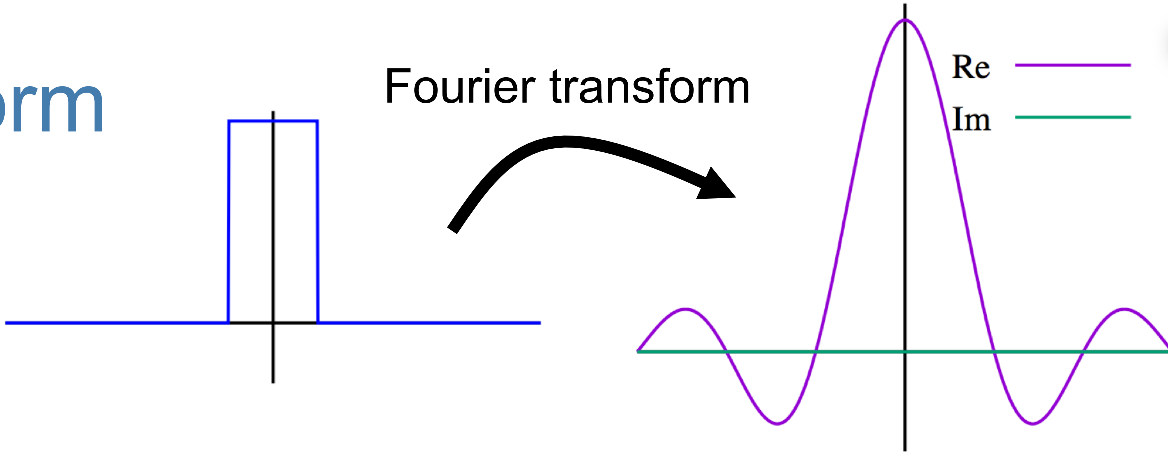


Time-domain signal

Frequency domain representation

Image from: https://en.wikipedia.org/wiki/Fourier_transform

Fourier Transform



Fourier transform: $X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$

Exists if $\int_{-\infty}^{\infty} |x(t)|dt < \infty$

Inverse transform: $\mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df$

Common terminology

Amplitude spectrum: $|X(f)|$

Phase spectrum: $\arg\{X(f)\}$

Fourier Transform – Examples

Complex exponential: $x(t) = e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j\sin(2\pi f_0 t)$

$$X(f) = \delta(f - f_0)$$

Cosine:

$$x(t) = \cos(2\pi f_0 t)$$

$$X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Sine:

$$x(t) = \sin(2\pi f_0 t)$$

$$X(f) = \frac{1}{j2}\delta(f - f_0) - \frac{1}{j2}\delta(f + f_0)$$

Rectangle pulse:

$$x(t) = u(t + 1) - u(t - 1)$$

$$X(f) = \mathrm{sinc}(f) = \frac{\sin(\pi f)}{\pi f}$$

Fourier Transform – Properties

Let $A(f) = \mathcal{F}\{a(t)\}$ and $B(f) = \mathcal{F}\{b(t)\}$

Convolution → Product:

$$\mathcal{F}\{(a * b)(t)\} = A(f)B(f)$$

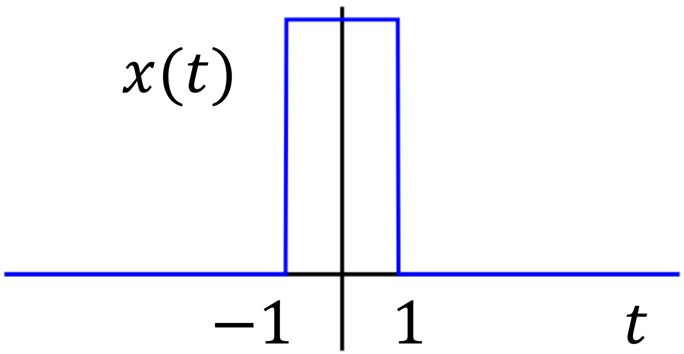
Product → Convolution:

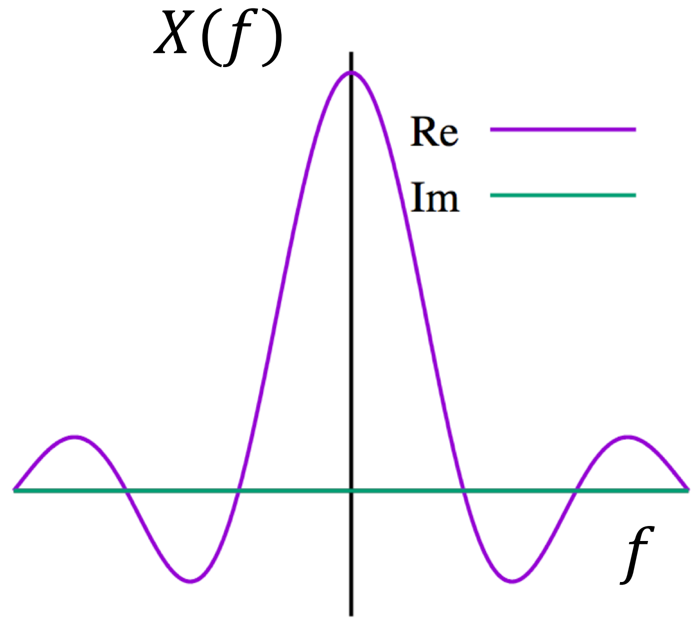
$$\mathcal{F}\{a(t)b(t)\} = (A * B)(f)$$

Time shift → Phase shift:

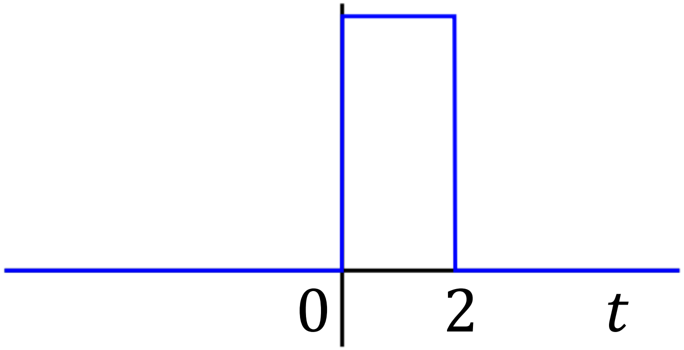
$$\mathcal{F}\{a(t-\tau)\} = A(f)e^{-j2\pi f\tau}$$

Properties – Example





$$y(t) = x(t-1)$$



$$Y(f) = X(f)e^{-j2\pi f}$$

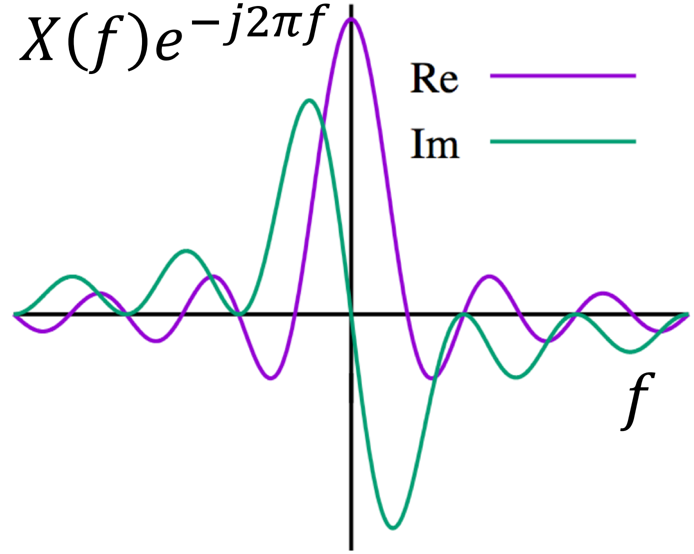


Image from: https://en.wikipedia.org/wiki/Fourier_transform

Example – Baseband to Passband

Recall:

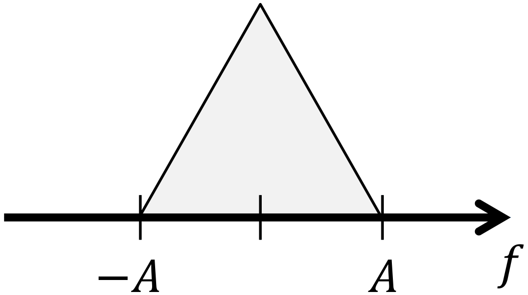
$$\mathcal{F}\{x(t)\} = X(f)$$

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

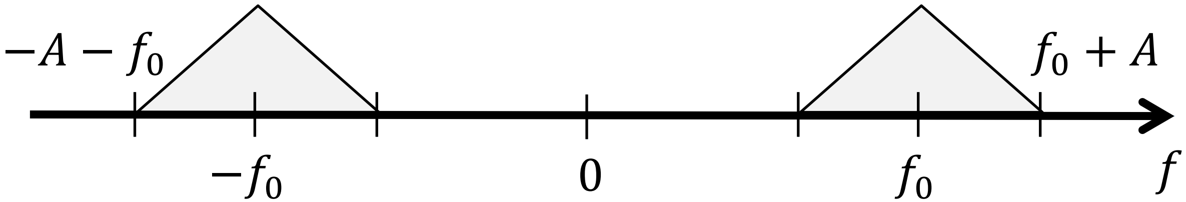
Consequence:

$$\mathcal{F}\{x(t)\cos(2\pi f_0 t)\} = \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$$

Baseband: $X(f)$



Passband: $\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$



Frequency Response of LTI System

Theorem: Let $x(t)$ be the input to an energy-free LTI system with impulse response $h(t)$, then the output of the system is

$$y(t) = (x * h)(t).$$

Definition: $H(f) = \mathcal{F}\{h(t)\}$ is called the frequency response.

Only exists for stable systems: $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

Property: The input and output of LTI systems are related as