TSKS21 Signaler, information & bilder

Föreläsning 9

Avslutning sampling och PAM: samplingsteoremet

Kvantisering och brus

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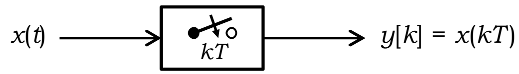
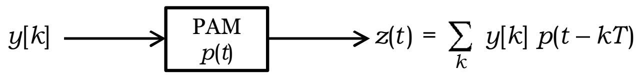
Linjära avbildningar

Sampling:

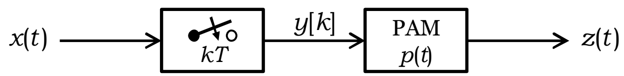


Pulsamplitudmodulering:
(PAM)

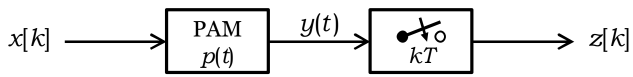


Rekonstruktion:

$t \to k \to t$:



$k \to t \to k$:



Sampling – Frekvensdomänen

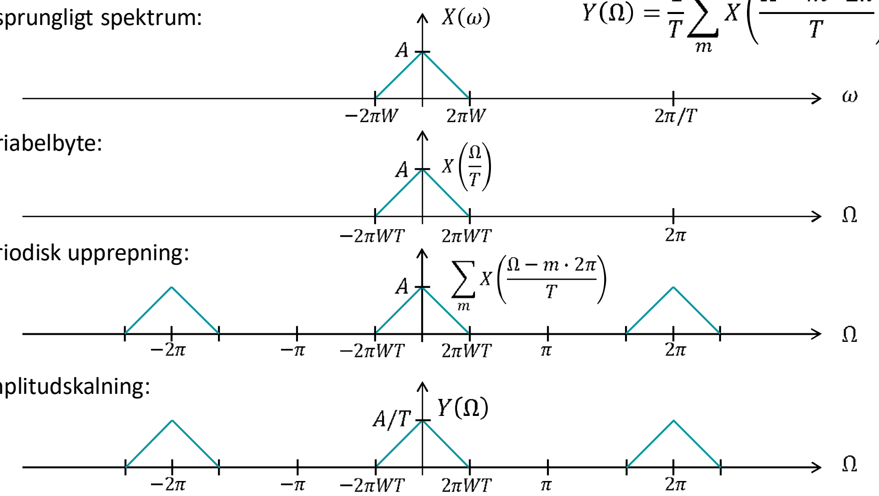
$$Y(\Omega) = \frac{1}{T}\sum_m X\left(\frac{\Omega - m\cdot 2\pi}{T}\right)$$

Ursprungligt spektrum:

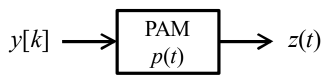
Variabelbyte:

Periodisk upprepning:

Amplitudskalning:



PAM - PulsAmplitudModulering



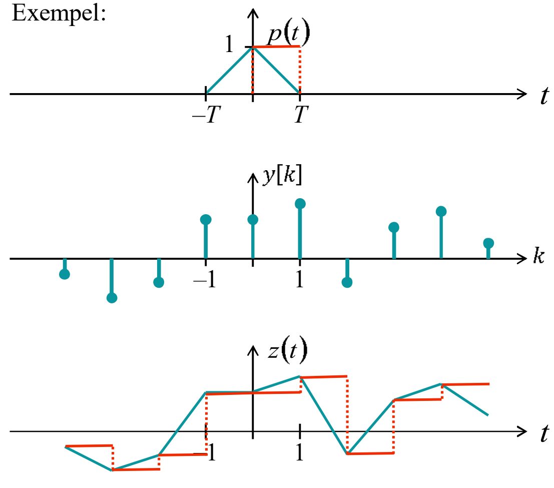
Tidsdomänen:

$$z(t) = \sum_k y[k]\, p(t - kT)$$

Frekvensdomänen:

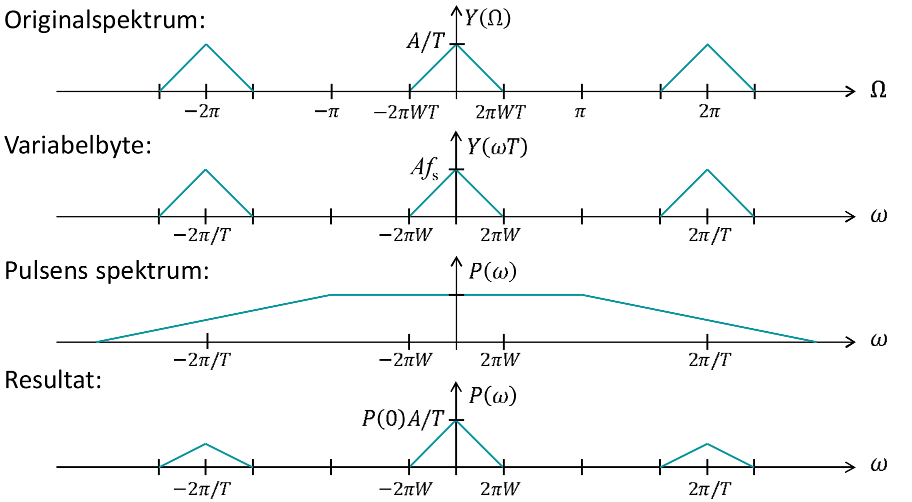
$$Z(\omega) = P(\omega)Y[\omega T]$$

Exempel:

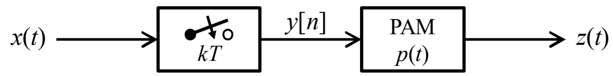


PAM – Frekvensdomänen

$$Z(\omega) = P(\omega)Y[\omega T]$$



Samplingsteoremet



Samplingsteoremet:

Betrakta en signal $x(t)$, med spektrum $X(\omega)$ och $X(\omega) = 0$ för $|\omega| \geq \omega_0$. Om $x(t)$ samplas med samplingsvinkelfrekvens ω_s, så kan $x(t)$ rekonstrueras utan fel från den samplade signalen om $\omega_s \geq 2\omega_0$ gäller.

Detta betyder:

Det finns en pulsform $p(t)$, så att $x(t)$ kan skrivas som

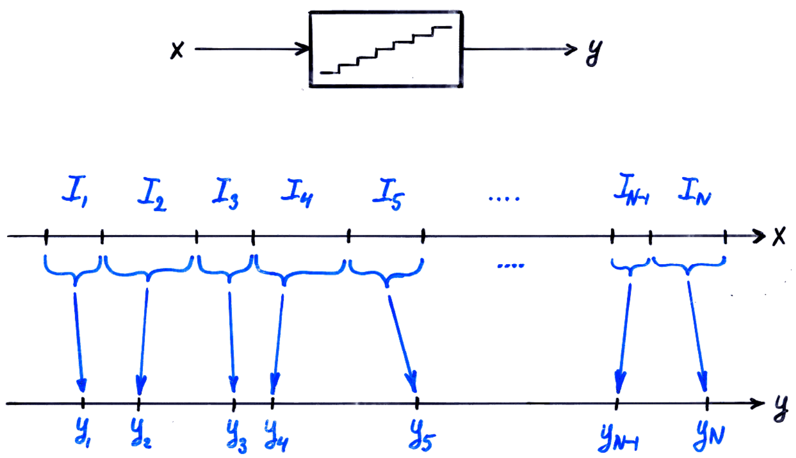
$$x(t) = \sum_k x(kT)\, p(t - kT)$$

om $\omega_s \geq 2\omega_0$ gäller, med $\omega_s = 2\pi/T$.

Detta gäller för:

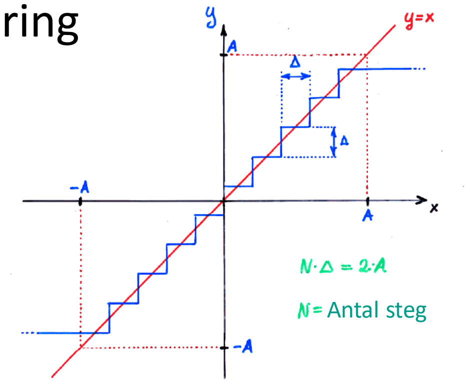
Ideal rekonstruktion: $p(t) = \mathrm{sinc}(t/T)$

Principen för Kvantisering

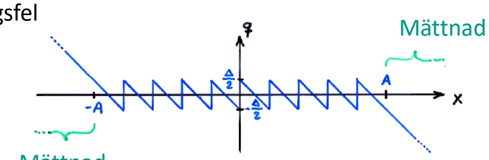


Likformig Kvantisering

$$y = g(x) = \begin{cases} A - \frac{\Delta}{2}, & x > A \\ \frac{\Delta}{2} + \left\lfloor \frac{x}{\Delta} \right\rfloor \cdot \Delta, & |x| \leq A \\ -A + \frac{\Delta}{2}, & x < -A \end{cases}$$

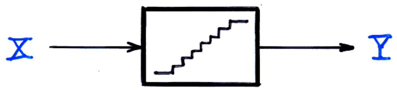


Kvantiseringsfel



$q = y - x = g(x) - x.$

Kvantiseringsdistorsion 1(2)



The error: $Q = Y - X = g(X) - X$

Quantization distortion:

$$P_Q = E\{Q^2\} = E\{(g(X) - X)^2\} = \int_{-\infty}^{\infty} (g(x) - x)^2 \cdot f_X(x)\, dx$$

Assumptions:

1. No saturation: $f_X(x) = 0$ for $|x| \geq A$
2. Nice distribution: $f_X(x)$ continuous for $|x| < A$
3. Small Δ: $f_X(x)$ approx. const. in intervals of length Δ.

Kvantiseringsdistorsion 2(2)

$$P_Q = \int_{-A}^{A} (g(x) - x)^2 f_X(x)\, dx = \sum_{k=1}^{N} \int_{y_k - \frac{\Delta}{2}}^{y_k + \frac{\Delta}{2}} (y_k - x)^2 \cdot f_X(x)\, dx = \Big/ \begin{matrix} u = x - y_k \\ du = dx \end{matrix} \Big/$$

$$= \sum_{k=1}^{N} \int_{-\Delta/2}^{\Delta/2} u^2 f_X(y_k + u)\, du \underset{\substack{\uparrow \\ \Delta \text{ small} \\ u \text{ small}}}{\approx} \sum_{k=1}^{N} \int_{-\Delta/2}^{\Delta/2} u^2 \cdot f_X(y_k)\, du$$

$$= \sum_{k=1}^{N} f_X(y_k) \underbrace{\int_{-\Delta/2}^{\Delta/2} u^2 du}_{= \Delta^3/12} = \frac{\Delta^2}{12} \sum_{k=1}^{N} \Delta \cdot f_X(y_k) \approx \frac{\Delta^2}{12} \underbrace{\sum_{k=1}^{N} Pr\{X \in I_k\}}_{=1} = \frac{\Delta^2}{12}$$

Error distribution: Approx. uniformly distr. over $[-\frac{\Delta}{2}, \frac{\Delta}{2})$

Generally without saturation:

$P_Q \leq \frac{\Delta^2}{4}$ since $|Q| \leq \frac{\Delta}{2}$.

SDR för likformig kvantisering

SDR – Signal-till-Distorsions-Förhållande (Ratio)

Fortfarande begränsad till [-*A*,*A*] och med tillräckligt snäll fördelning.

$$P_Q = \mathrm{E}\{Q^2\} = \int_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta} dq = \frac{\Delta^2}{12} = \frac{A^2}{3N^2} \quad \Rightarrow \quad \mathrm{SDR} = \frac{P_X}{P_Q} = \frac{3P_X}{A^2} N^2 = \frac{3P_X}{A^2} 2^{2n}$$

$$\mathrm{SDR}_{\mathrm{dB}} = 10\log_{10}(\mathrm{SDR}) = 10\log_{10}\left(\frac{3P_X}{A^2}\right) + n \cdot 20\log_{10}(2) \approx 10\log_{10}\left(\frac{3P_X}{A^2}\right) + 6n.$$

Exempel: Likformig fördelning över [-*A*,*A*].

$$P_X = \mathrm{E}\{X^2\} = \int_{-A}^{A} x^2 \frac{1}{2A} dx = \frac{A^2}{3},$$

$$\mathrm{SDR}_{\mathrm{dB}} \approx 10\log_{10}\left(\frac{3A^2/3}{A^2}\right) + 6n = 10\log_{10}(1) + 6n = 6n$$

SDR för likformig kvantisering med mättnad

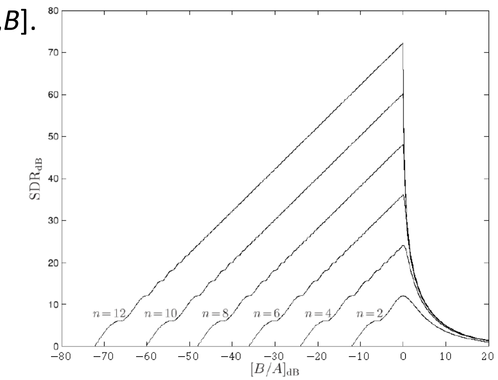
Likformig fördelning över [-*B*,*B*].

$$P_X = \mathrm{E}\{X^2\} = \int_{-B}^{B} x^2 \frac{1}{2B} dx = \frac{B^2}{3}.$$

Q och S okorrelerade:

$$P_{Q+S} = P_Q + P_S.$$

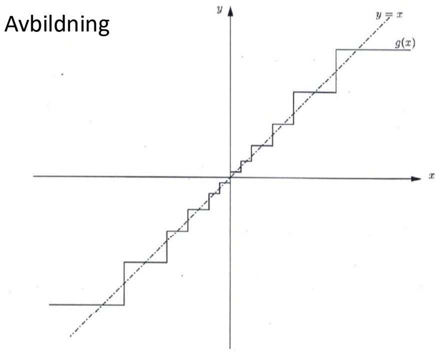
$$\mathrm{SDR} = \frac{P_X}{P_Q + P_S},$$



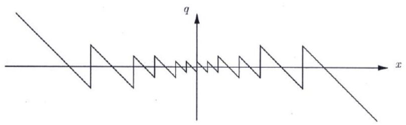
$$P_{Q+S} = \begin{cases} \frac{(2k+1)(\Delta/2)^3 + (B-(2k+1)\Delta/2)^3}{3B}, & k\Delta \leq B < (k+1)\Delta,\ k \in \{0, 1, ..., N/2-2\} \\ \frac{A-\Delta/2}{B} \cdot \frac{\Delta^2}{12} + \frac{(B-A+\Delta/2)^3}{3B} & B \geq A - \frac{\Delta}{2} \end{cases}$$

Olikformig kvantisering

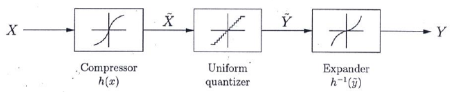
Avbildning

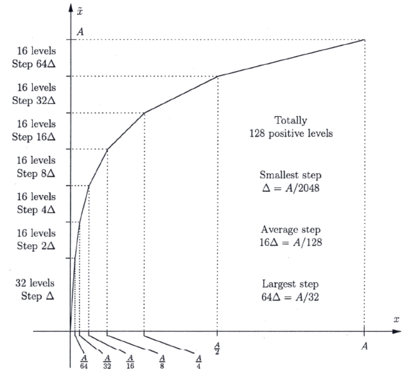


Kvantiseringsfel



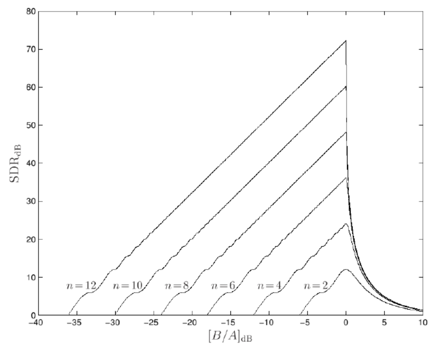
Implementering



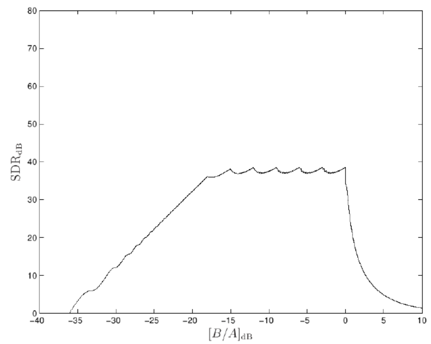


SDR för olikformig kvantisering

Likformig kvantisering

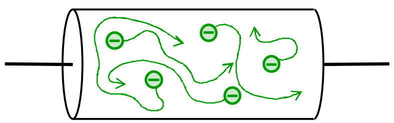


Olikformig kvantisering



Termiskt brus 1(3) Fysiken bakom

Ett motstånd:



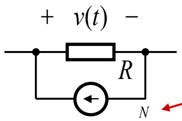
Termiska rörelser hos elektroner

$\Rightarrow$ Slumpmässiga lokala strömmar

$\Rightarrow$ Slumpmässiga lokala spänningar

$\Rightarrow$ Slumpmässig total spänning

Modell:



$v(t) \quad R \cdot i(t)$

Enorm N

$$i(t) = \sum_{k=1}^{N} i_k(t)$$

$\Rightarrow$ $i(t)$ gaussisk

$\Rightarrow$ $v(t)$ gaussisk

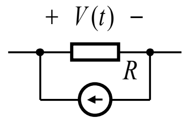
Korta pulser, nästan enhetsimpulser

$\Rightarrow$ $i(t_1)$ & $i(t_2)$ nästan oberoende för $t_1 \neq t_2$

Vitt gaussiskt brus

Termiskt brus 2(3) – Spektraltäthet

Fortsatt modell:



Vitt gaussiskt brus med

$$R_V(f) = \frac{N_0}{2} = 2kTR$$

$k \approx 1.38 \cdot 10^{-23}$ J/K (Boltzmanns konstant)

T = Absolut temperatur i Kelvin.

R = Resistans i Ohm.

$h \approx 6.63 \cdot 10^{-34}$ Js (Plancks konstant)

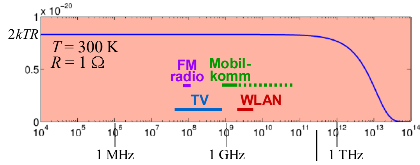
Mer exakt modell:

Gaussiskt brus med

$$R_V(f) = \frac{2Rh|f|}{e^{h|f|/kT} - 1}$$

Notera:

$$R_V(f) \to 2kTR \quad \text{när} \quad f \to 0$$



Reglerat radiospektrum

Termiskt brus 3(3) – Hur stort är stort?

Kol

Densitet: 1.5 kg/dm^3
#nukleoner: 12
Atommassa: $12 \cdot 1.7 \cdot 10^{-27}$ kg/atom

$$\Rightarrow \frac{1.5}{12 \cdot 1.7 \cdot 10^{-27}} = 7.4 \cdot 10^{25} \text{ atomer/dm}^3$$

Motstånd

Tvärsnittsarea: 0.25 mm^2
Längd: 2 mm

$\Rightarrow$ Volym 0.5 mm^3 = $0.5 \cdot 10^{-6}$ dm^3

$$\Rightarrow 0.5 \cdot 10^{-6} \cdot 7.4 \cdot 10^{25} \approx 3.8 \cdot 10^{19} \text{ kolatomer}$$

Kol har 4 valenselektroner/atom

$$\Rightarrow \text{ Totalt } \quad 4 \cdot 3.8 \cdot 10^{19} \approx 1.5 \cdot 10^{20} \text{ valenselektroner.}$$

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