TSDT14 Signal Theory

Lecture 7
Analog Modulation

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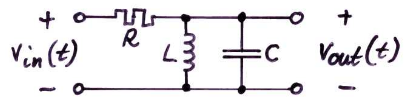
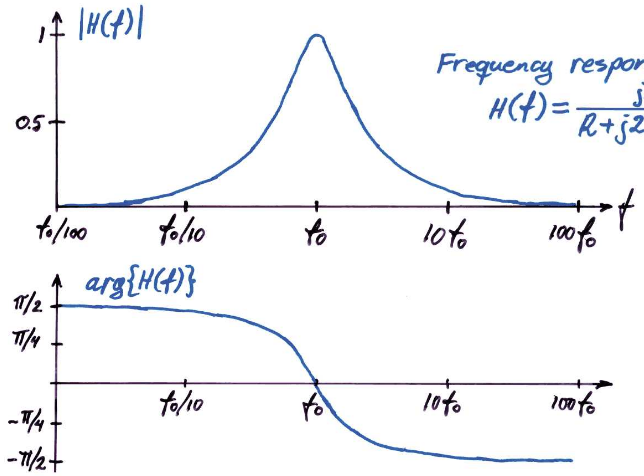
A Simple BP Filter



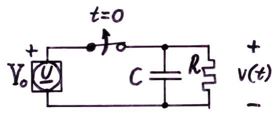


Frequency response:

$$H(f) = \frac{j2\pi fL}{R + j2\pi fL - 4\pi^2 f^2 RLC}$$

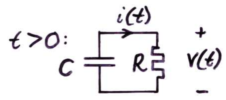
$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Discharging a Capacitor



Initial condition

$v(0) = V_0$



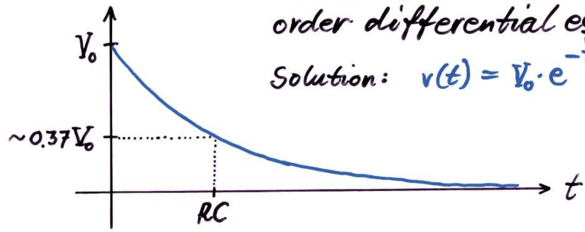
$v(t) = R \cdot i(t) = -RC\frac{d}{dt}v(t)$

$i(t) = -C\frac{d}{dt}v(t)$

$\Rightarrow RC\frac{d}{dt}v(t) + v(t) = 0$

This is the standard example of a first order differential equation.

Solution: $v(t) = V_0 \cdot e^{-t/RC}$ for $t \geq 0$



Amplitude Modulation

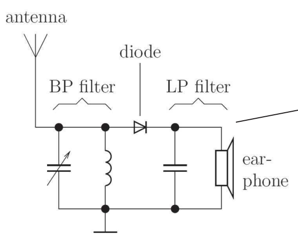
- The first technique used for radio broadcasts.
- A linear modulation technique.
- Simple to analyze.
- Simple demodulation.
- Noise sensitive.

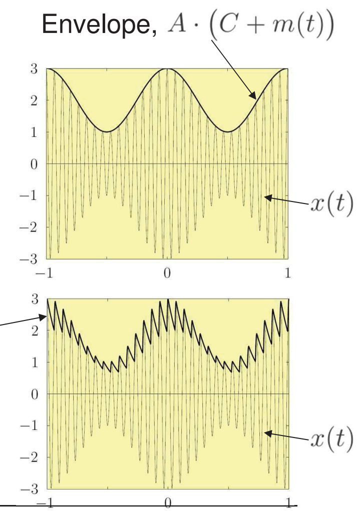
Amplitude Modulation – Deterministic Case

Standard AM:

$$x(t) = A \cdot \big(C + m(t)\big) \cos(2\pi f_c t)$$

Crystal receiver, an envelope detector, first demodulator of standard AM:

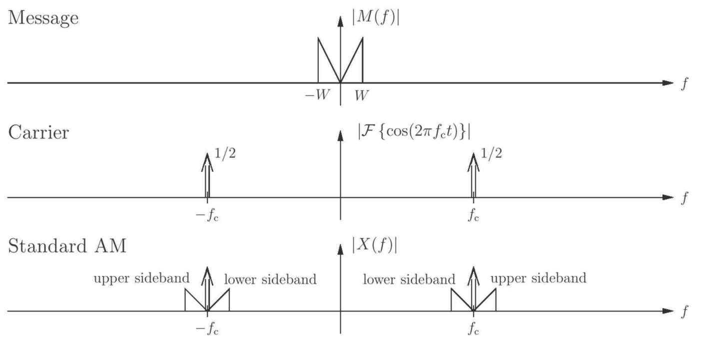




Spectrum of Standard AM

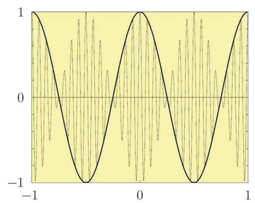
$$x(t) = A \cdot \big(C + m(t)\big) \cos(2\pi f_c t)$$

$$X(f) = \mathcal{F}\{AC \cos(2\pi f_c t)\} + \mathcal{F}\{A\, m(t) \cos(2\pi f_c t)\}$$
$$= \frac{AC}{2}\big(\delta(f-f_c) + \delta(f+f_c)\big) + \frac{A}{2}\big(M(f-f_c) + M(f+f_c)\big),$$

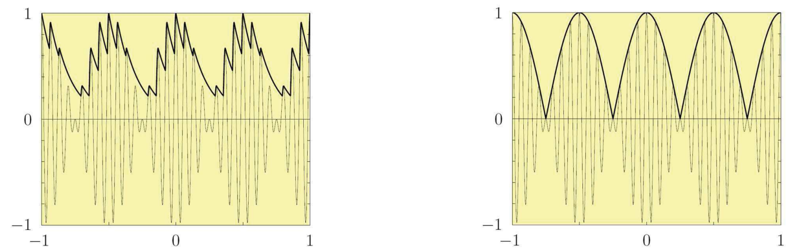


AM-SC – Suppressed Carrier

AM-SC: $x(t) = A\, m(t) \cos(2\pi f_c t)$



Demodulating AM-SC with an envelope detector:

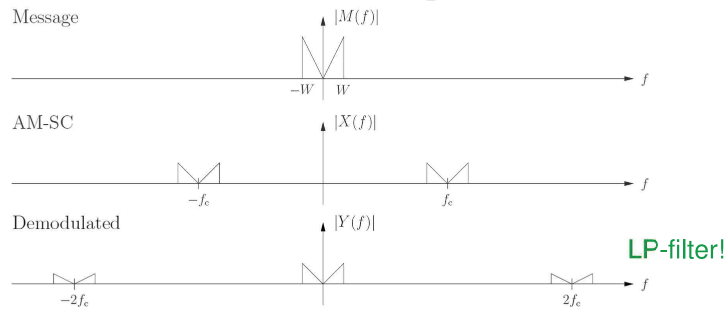


Spectrum of AM-SC – and Demodulation

$$x(t) = A\, m(t) \cos(2\pi f_c t) \quad \Rightarrow \quad X(f) = \frac{A}{2}\big(M(f-f_c) + M(f+f_c)\big)$$

Coherent demodulation:

$$y(t) = 2\, x(t) \cos(2\pi f_c t) = 2A\, m(t) \cos^2(2\pi f_c t) = A\, m(t)\big(1 + \cos(4\pi f_c t)\big),$$
$$Y(f) = A\big(X(f-f_c) + X(f+f_c)\big) = A \cdot M(f) + \frac{A}{2}\big(M(f-2f_c) + M(f+2f_c)\big).$$

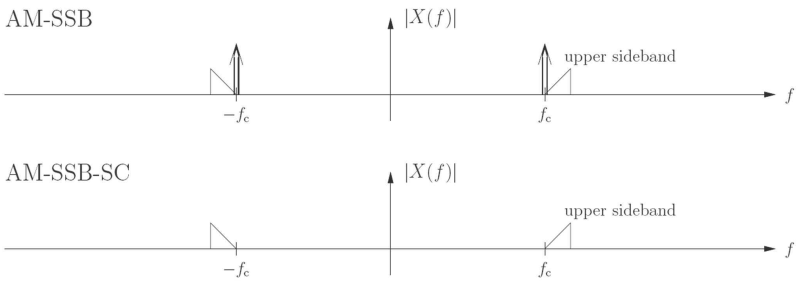


AM-SSB – Single Side-Band

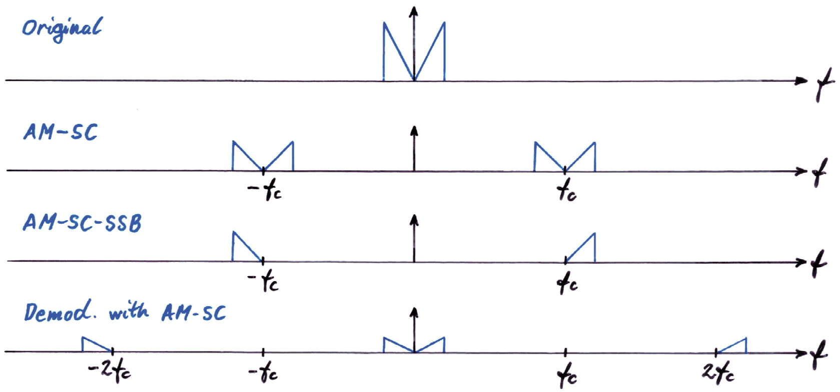
AM (SC) uses twice as much bandwidth as needed.

Each sideband contains all the information.

Filter out one of the sidebands.



Demodulate SSB



LP-filter again!

AM of Stochastic Processes 1(3)

As for deterministic signals: $X(t) = A \cdot (C + M(t)) \cos(2\pi f_c t)$

Question: Is $X(t)$ stationary in any sense if $M(t)$ is stationary?

ACF:

$$r_X(t, t+\tau) =$$
$$= \mathrm{E}\left\{A(C + M(t)) \cos(2\pi f_c t)\, A(C + M(t+\tau)) \cos(2\pi f_c (t+\tau))\right\},$$
$$= A^2 \mathrm{E}\left\{C^2 + C(M(t) + M(t+\tau)) + M(t)M(t+\tau)\right\} \cos(2\pi f_c t) \cos(2\pi f_c (t+\tau))$$
$$= \frac{A^2}{2}\left(C^2 + 2Cm_M + r_M(\tau)\right)\Big(\cos(2\pi f_c (2t+\tau)) + \cos(2\pi f_c \tau)\Big),$$

Dependent on t. Non-stationary.

AM of Stochastic Processes 2(3)

Adjust the situation: $X(t) = A \cdot (C + M(t)) \cos(2\pi f_c t + \Psi)$

Question: Is $X(t)$ stationary now?

Unif. on $[0,2\pi)$.
Indep of $X(t)$.

Mean:
$$m_X(t) = \mathrm{E}\{X(t)\} = \mathrm{E}\{A \cdot (C + M(t)) \cos(2\pi f_c t + \Psi)\}$$
$$= \mathrm{E}\{A \cdot (C + M(t))\} \cdot \mathrm{E}\{\cos(2\pi f_c t + \Psi)\},$$

Note:
$$\mathrm{E}\{\cos(2\pi f_c t + \Psi)\} = \int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi f_c t + \psi)\, d\psi = 0,$$

Result: $m_X(t) = 0$

OK! Independent of t.
What about the ACF?

AM of Stochastic Processes 3(3)

Still this situation: $X(t) = A \cdot \big(C + M(t)\big) \cos(2\pi f_c t + \Psi)$

ACF:

$$r_X(t,t+\tau) = \mathrm{E}\left\{A\cdot\big(C+M(t)\big)\cos(2\pi f_c t+\Psi)\, A\cdot\big(C+M(t+\tau)\big)\cos\big(2\pi f_c(t+\tau)+\Psi\big)\right\},$$
$$= A^2\mathrm{E}\left\{C^2 + C\big(M(t)+M(t+\tau)\big) + M(t)M(t+\tau)\right\}\cdot$$
$$\cdot\,\mathrm{E}\left\{\cos(2\pi f_c t+\Psi)\cos\big(2\pi f_c(t+\tau)+\Psi\big)\right\},$$
$$= \frac{A^2}{2}\big(C^2 + 2Cm_M + r_M(\tau)\big)\Big(\mathrm{E}\left\{\cos\big(2\pi f_c(2t+\tau)+2\Psi\big)\right\} + \cos(2\pi f_c\tau)\Big),$$

Note: $\mathrm{E}\left\{\cos\big(2\pi f_c(2t+\tau)+2\Psi\big)\right\} = \int_0^{2\pi} \frac{1}{2\pi}\cos\big(2\pi f_c(2t+\tau)+2\psi\big)\,d\psi = 0.$

Result: $r_X(t,t+\tau) = \frac{A^2}{2}\big(C^2 + 2Cm_M + r_M(\tau)\big)\cos(2\pi f_c\tau),$

OK! Independent of *t*. Stationary in the wide sense.

PSD of AM

Often we have: $m_M = 0.$

ACF: $r_X(\tau) = \frac{A^2}{2}\big(C^2 + r_M(\tau)\big)\cos(2\pi f_c\tau)$

PSD: $R_X(f) = \frac{A^2C^2}{4}\big(\delta(f+f_c)+\delta(f-f_c)\big) + \frac{A^2}{4}\big(R_M(f+f_c)+R_M(f-f_c)\big)$

Power: $P_X = r_X(0) = \frac{A^2}{2}\big(C^2 + r_M(0)\big)\cos(0) = \frac{A^2}{2}\big(C^2+r_M(0)\big) = \frac{A^2}{2}\big(C^2+P_M\big)$

For AM-SC:

$$r_X(\tau) = \frac{A^2}{2} r_M(\tau)\cos(2\pi f_c\tau),$$
$$R_X(f) = \frac{A^2}{4}\big(R_M(f+f_c)+R_M(f-f_c)\big),$$
$$P_X = \frac{A^2}{2}P_M.$$

Coherent Demodulation of AM-SC

Coherent demodulation: $Y(t) = 2X(t)\cos(2\pi f_c t + \Psi)$

Same Ψ as for modulation.

Demodulated signal:

$$Y(t) = 2A\,M(t)\cos^2(2\pi f_c t+\Psi) = A\,M(t)\big(1+\cos(4\pi f_c t + 2\Psi)\big),$$

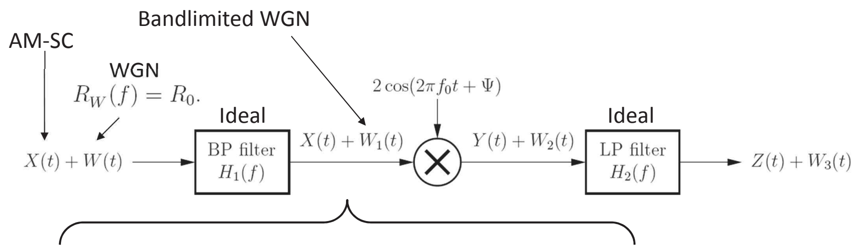
Result:

$$r_Y(\tau) = A^2 r_M(\tau) + \frac{A^2}{2} r_M(\tau)\cos(4\pi f_c\tau),$$
$$R_Y(f) = A^2 R_M(f) + \frac{A^2}{4}\big(R_M(f+2f_c)+R_M(f-2f_c)\big),$$
$$P_Y = \frac{3A^2}{2}P_M.$$

After ideal LP-filter:

$$r_Z(\tau) = A^2\, r_M(\tau), \qquad R_Z(f) = A^2 R_M(f), \qquad P_Z = A^2 P_M.$$

Noise Analysis of AM-SC 1(4)



$$H_1(f) = \begin{cases} 1, & \big||f|-f_c\big| < B, \\ 0, & \text{elsewhere.}\end{cases}$$

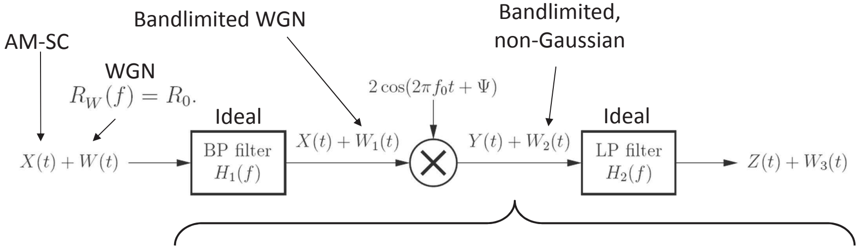
$$R_{W_1}(f) = |H_1(f)|^2 R_W(f) = \begin{cases} R_0, & \big||f|-f_c\big| < B, \\ 0, & \text{elsewhere,}\end{cases}$$

$$P_{W_1} = \int_{-\infty}^{\infty} R_{W_1}(f)\,df = 2\int_{f_c-B}^{f_c+B} R_0\,df = 4BR_0.$$

Input SNR

$$\frac{P_X}{P_{W_1}} = \frac{A^2P_M/2}{4BR_0} = \frac{A^2P_M}{8BR_0}$$

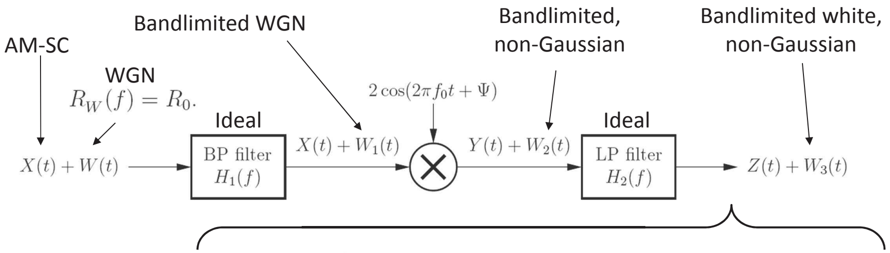
Noise Analysis of AM-SC 2(4)



$$W_2(t) = 2W_1(t)\cos(2\pi f_c t + \Psi)$$

$$R_{W_2}(f) = R_{W_1}(f + f_c) + R_{W_1}(f - f_c) = \begin{cases} 2R_0, & |f| < B, \\ R_0, & \big||f| - 2f_c\big| < B, \\ 0, & \text{elsewhere.} \end{cases}$$

Noise Analysis of AM-SC 3(4)



$$H_2(f) = \begin{cases} 1, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$$

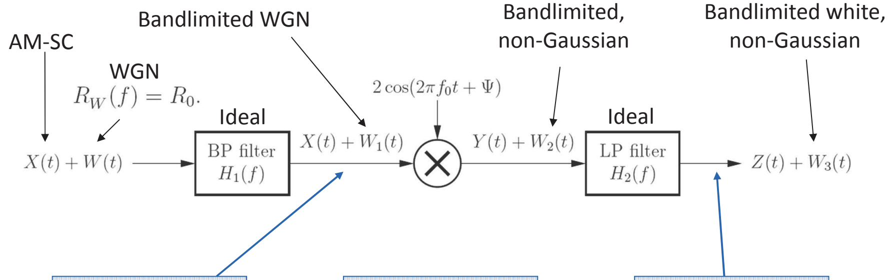
$$R_{W_3}(f) = \left|H_2(f)\right|^2 R_{W_2}(f) = \begin{cases} 2R_0, & |f| < B, \\ 0, & \text{elsewhere,} \end{cases}$$

$$P_{W_3} = \int_{-\infty}^{\infty} R_{W_3}(f)\, df = \int_{-B}^{B} 2R_0\, df = 4BR_0.$$

Output SNR

$$\frac{P_Z}{P_{W_3}} = \frac{A^2 P_M}{4BR_0}$$

Noise Analysis of AM-SC 4(4)



Input SNR

$$\frac{P_X}{P_{W_1}} = \frac{A^2 P_M}{8BR_0}$$

SNR Relation

$$\frac{P_Z}{P_{W_3}} = 2\frac{P_X}{P_{W_1}}$$

Output SNR

$$\frac{P_Z}{P_{W_3}} = \frac{A^2 P_M}{4BR_0}$$

Twice the bandwidth – Twice the SNR.

Angle Modulation

- The major modulation techniques used in radio broadcasts today are examples of angle modulation.
  - FM – Frequency Modulation
  - PM – Phase Modulation
- Nonlinear modulation techniques.
- Complicated to analyze.
- Still fairly simple demodulation.
- Less sensitive to noise than AM.

Angle Modulation – Modulation Indices

Angle modulation: $x(t) = A \cdot \cos\left(2\pi f_c t + \underbrace{\phi\{m(t)\}}_{\text{Momentary phase}}\right)$

Phase deviation: $\phi_d(t) = \phi\{m(t)\},$ for mean 0.

Phase modulation index: $\mu_p = \phi_{d,\max} = \max|\phi_d(t)|$

Momentary frequency: $f_{\text{mom}}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt}\left(2\pi f_c t + \phi\{m(t)\}\right) = f_c + \frac{1}{2\pi} \cdot \frac{d}{dt}\phi\{m(t)\}$

Frequency deviation: $f_d(t) = f_{\text{mom}}(t) - f_c = \frac{1}{2\pi} \cdot \frac{d}{dt}\phi\{m(t)\}$

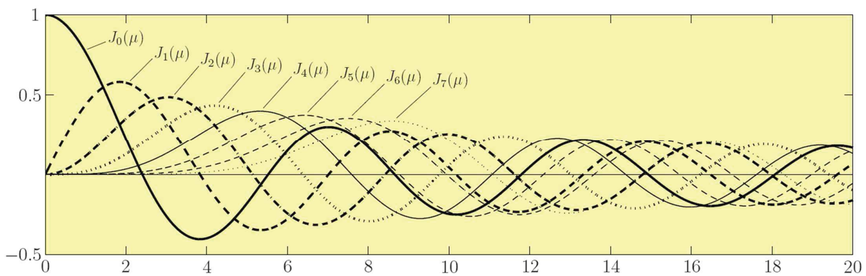
Frequency modulation idx: $\mu_f = \frac{f_{d,\max}}{B}$ $\qquad f_{d,\max} = \max|f_d(t)|$

Spectrum of Angle Modulation 1(2)

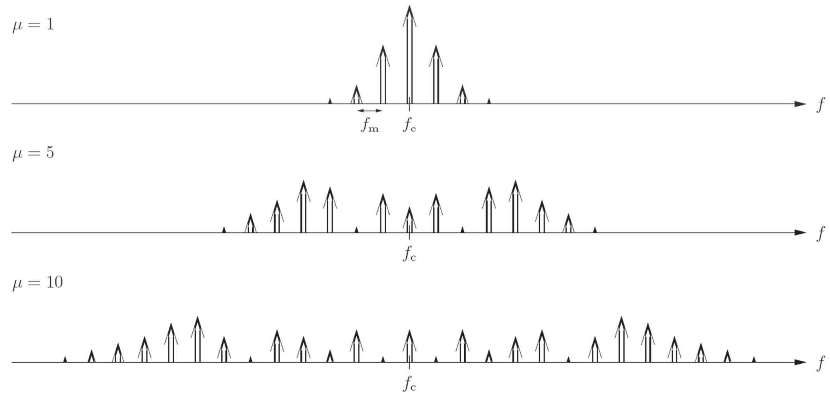
Example: $x(t) = A \cdot \cos\left(2\pi f_c t + \mu \sin(2\pi f_m t)\right) = \sum_{n=-\infty}^{\infty} A \cdot J_n(\mu) \cos\left(2\pi(f_c + n f_m)t\right)$

Spectrum: $X(f) = \sum_{n=-\infty}^{\infty} \frac{A \cdot J_n(\mu)}{2}\Big(\delta(f + f_c + n f_m) + \delta(f - f_c - n f_m)\Big).$

Bessel functions of the first kind of order *n*: $J_n(\mu) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{\mu}{2}\right)^{n+2k}$



Spectrum of Angle Modulation 2(2)



Carsons rule: Bandwidth $\approx 2(\mu+1) f_m = 2\left(1 + \frac{1}{\mu}\right) f_{d,\max}.$

PM – Phase Modulation

Momentary phase: $\phi\{m(t)\} = a \cdot m(t),$

Signal: $x(t) = A \cdot \cos\left(2\pi f_c t + a \cdot m(t)\right).$

Momentary frequency: $f_{\text{mom}}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt}\left(2\pi f_c t + a \cdot m(t)\right) = f_c + \frac{a}{2\pi} \cdot \frac{d}{dt} m(t),$

Frequency deviation: $f_d(t) = \frac{a}{2\pi} \cdot \frac{d}{dt} m(t).$

Peak frequency dev.: $f_{d,\max} = \frac{a}{2\pi} \cdot \max\left|\frac{d}{dt} m(t)\right|$

Frequency mod. idx: $\mu_f = \frac{a}{2\pi B} \cdot \max\left|\frac{d}{dt} m(t)\right|$

FM – Frequency Modulation

Momentary phase: $\phi\{m(t)\} = a \int m(t)\,dt, \qquad \phi\{m(t)\} = a \int_{t_0}^{t} m(\tau)\,d\tau,$

Signal: $x(t) = \cos\left(2\pi f_c t + a \int m(t)\,dt\right)$

Momentary frequency: $f_{\text{mom}}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt}\left(2\pi f_c t + a \int m(t)\,dt\right) = f_c + \frac{a}{2\pi} \cdot m(t).$

Frequency deviation: $f_d(t) = \frac{a}{2\pi} \cdot m(t).$

Peak frequency dev.: $f_{d,\max} = \frac{a}{2\pi} \cdot \max |m(t)|$

Frequency mod. idx: $\mu_f = \frac{a}{2\pi B} \cdot \max |m(t)|$

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Angle Modulation in the Time Domain

Phase Modulation

Frequency Modulation