TSKS01 Digital Communication
Lecture 2

Stochastic Processes, Noise Modeling, Pulse-Amplitude Modulation

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Multi-Dimensional Stochastic Variables

Distribution: $F_{X_1,\dots,X_N}(x_1,\dots,x_N) = \Pr\{X_1 \le x_1,\dots,X_N \le x_N\}$

Density: $f_{X_1,\dots,X_N}(x_1,\dots,x_N) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N} F_{X_1,\dots,X_N}(x_1,\dots,x_N)$

Vector notation: $\bar{X} = (X_1,\dots,X_N)$, $\bar{x} = (x_1,\dots,x_N)$, $F_{\bar{X}}(\bar{x})$, $f_{\bar{X}}(\bar{x})$

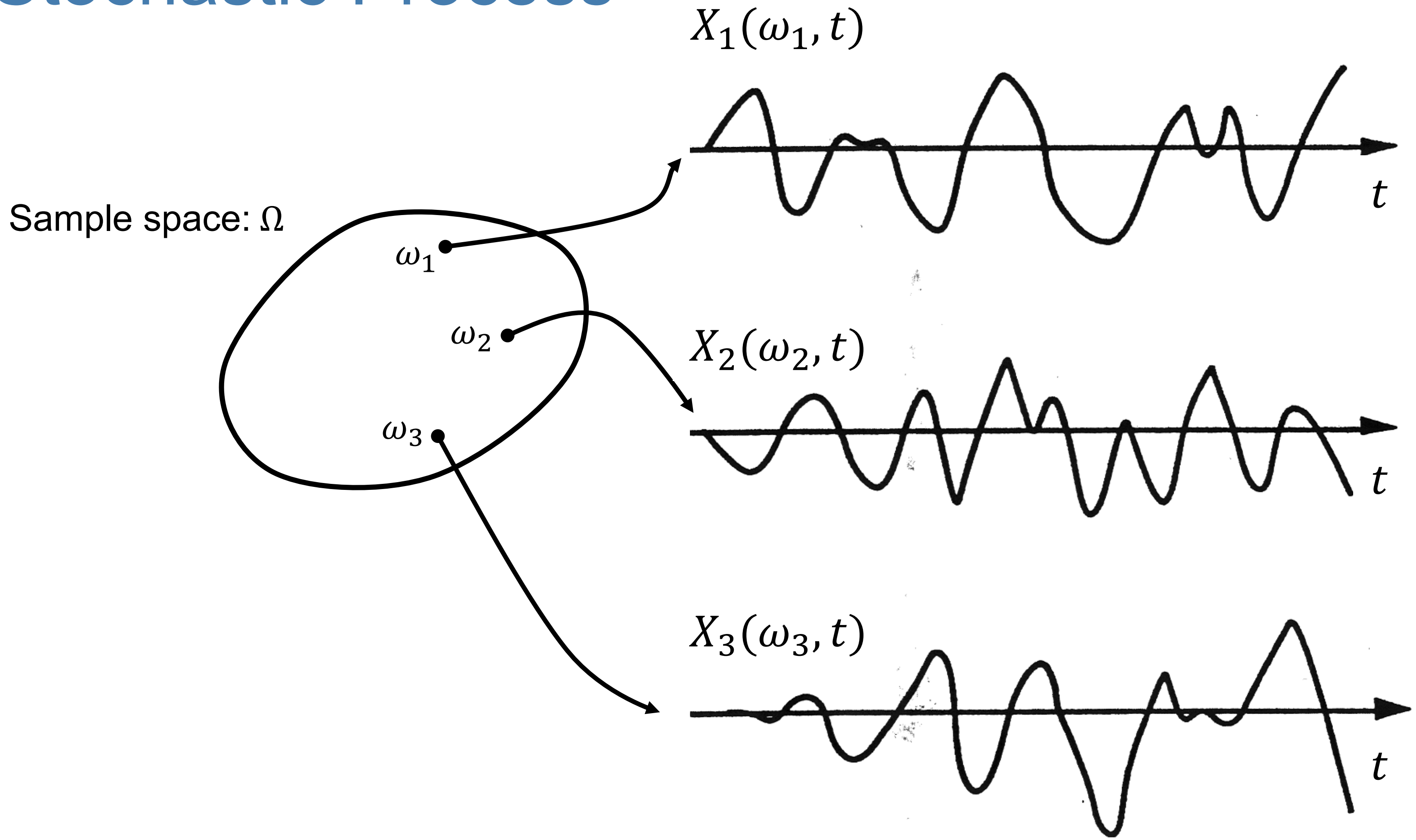
Example: $\bar{X} = (X_1,\dots,X_N)$ is *jointly Gaussian*, denoted as $N(\bar{m},\sigma^2 I)$, if

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^N \sigma^{2N}}} e^{-\frac{1}{2\sigma^2}(\bar{x}-\bar{m})(\bar{x}-\bar{m})^T}$$

where $\bar{m} = \mathrm{E}\{\bar{X}\}$ and $(\bar{x}-\bar{m})(\bar{x}-\bar{m})^T = \sum_i (x_i - m_i)^2$

There is a more general form with another matrix than $\sigma^2 I$

Stochastic Process



Stochastic process = Stochastic time-continuous signal ($N = \infty$)

Examples of Stochastic Processes

Example 1: Finite number of realizations:

$$X(t) = \sin(t + \phi), \qquad \phi \in \{0, \pi/2, \pi, 3\pi/2\}.$$

Example 2: Infinite number of realizations:

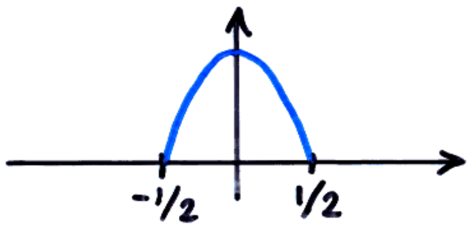
$$X(t) = A \cdot \sin(t), \qquad A \sim N(0,1).$$

Examples of Stochastic Processes cont'd

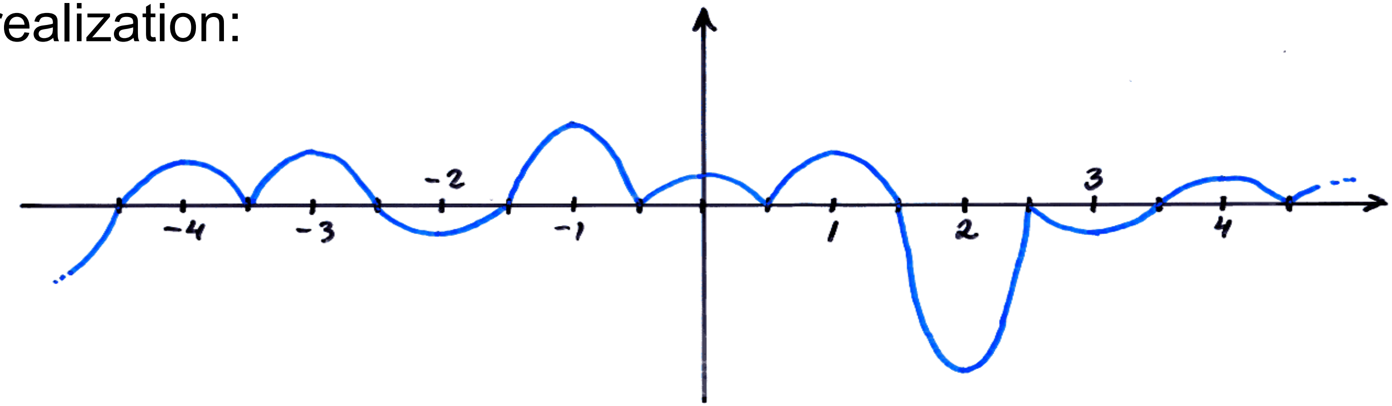
Example 3: Infinite number of realizations:

$$X(t) = \sum_k A_k g(t-k), \qquad g(t) = \begin{cases} \cos(\pi t), & |t| < 1/2 \\ 0, & \text{elsewhere} \end{cases}$$

Each A_k is independent and $N(0,1)$



One realization:



Sampling of Stochastic Process: N Time Instants

Vector notation

- Time instants: $\bar{t} = (t_1, \dots, t_N)$
- Stochastic variable: $X(\bar{t}) = (X(t_1), \dots, X(t_N))$
- Realizations: $\bar{x} = (x_1, \dots, x_N)$

N time instants

- Distribution: $F_{X(\bar{t})}(\bar{x}) = \Pr\{X(t_1) \le x_1, \dots, X(t_N) \le x_N\}$
- Density: $f_{X(\bar{t})}(\bar{x}) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N} F_{X(\bar{t})}(\bar{x})$

Sampling of stochastic process → Stochastic variable

Ensemble Averages

"Observe many realizations and make an average" – functions of time

Expectation (mean):

$$m_X(t) = E\{X(t)\}$$
$$= \int_{-\infty}^{\infty} x f_{X(t)}(x)dx$$

Quadratic mean (power):

$$E\{X^2(t)\} = \int_{-\infty}^{\infty} x^2 f_{X(t)}(x)dx$$

Variance

$$\sigma_{X(t)}^2 = E\left\{\left(X(t) - m_X(t)\right)^2\right\}$$
$$= E\{X^2(t)\} - m_X^2(t)$$

Auto-correlation function (ACF):

$$r_X(t_1, t_2) = E\{X(t_1)X(t_2)\} =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1 dx_2$$

Symmetry: $r_X(t_1, t_2) = r_X(t_2, t_1)$

Power: $r_X(t, t) = E\{X^2(t)\}$

Special case: A is stochastic var.

$$X(t) = g(t, A)$$

$$r_X(t_1, t_2) =$$
$$\int_{-\infty}^{\infty} g(t_1, A) g(t_2, A) f_A(a)\, da$$

Wide-Sense Stationarity

Definition: A stochastic process $X(t)$ is said to be *wide-sense stationary* (WSS) if

- Mean satisfies $m_X(t) = m_X(t + \Delta)$ for all Δ.
- ACF satisfies $r_X(t_1, t_2) = r_X(t_1 + \Delta, t_2 + \Delta)$ for all Δ.

Interpretation:

Constant mean, ACF only depends on time difference $\tau = t_1 - t_2$

Notation: Mean m_X

ACF $r_X(\tau)$

Gaussian Processes

Recall: $\bar{X} = (X_1, \dots, X_N)$ is *jointly Gaussian*, denoted as $N(\bar{m}, \sigma^2 I)$, if

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^N \sigma^{2N}}} e^{-\frac{1}{2\sigma^2}(\bar{x}-\bar{m})(\bar{x}-\bar{m})^T}$$

Definition: A stochastic process is called *Gaussian* if $X(\bar{t}) = (X(t_1), \dots, X(t_N))$ is jointly Gaussian for any $\bar{t} = (t_1, \dots, t_N)$

There is a more general form with another matrix than $\sigma^2 I$

Definition: A process with $r_X(\tau) = \text{constant} \cdot \delta(t)$ is called *white.*

Power-Spectral Density (PSD)

Definition: Fourier transform of the ACF of a WSS process:

$$R_X(f) = \mathcal{F}\{r_X(\tau)\} = \int_{-\infty}^{\infty} r_X(\tau) e^{-j2\pi f\tau} d\tau$$

Inverse:

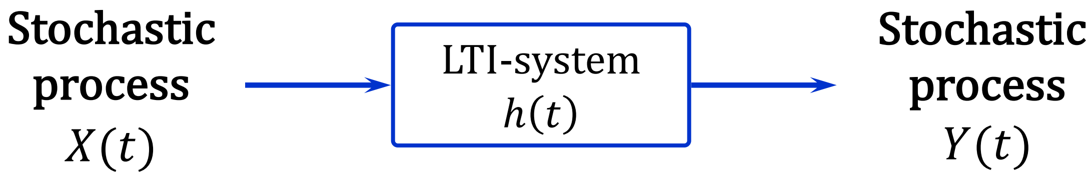
$$r_X(\tau) = \mathcal{F}^{-1}\{R_X(f)\} = \int_{-\infty}^{\infty} R_X(f) e^{j2\pi f\tau} df$$

Power:

$$E\{X^2(t)\} = r_X(0) = \int_{-\infty}^{\infty} R_X(f) df$$

One-sided PSD (of real-valued signal): $R_X(f) + R_X(-f)$ for $f \geq 0$

Filtering of Stochastic Process



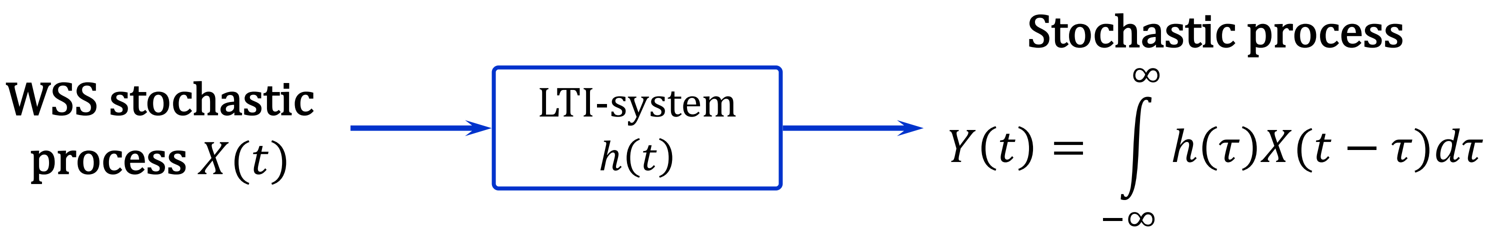
Input-output relation:

$$Y(t) = (X * h)(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$$

Requires stability: $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$

Holds regardless of stationarity

Filtering of WSS Stochastic Process



Notation: $H(f) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$

Mean: $m_Y(t) = m_X H(0)$

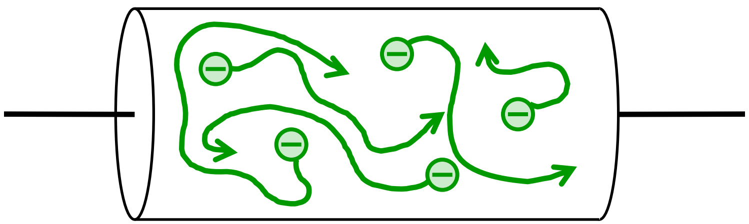
ACF: $r_Y(t_1, t_2) = (h * \tilde{h} * r_X)(\tau)$ where $\tilde{h}(t) = h(-t)$.

PSD: $R_Y(f) = \mathcal{F}\{r_Y(\tau)\} = H(f)H^*(f)R_X(f) = |H(f)|^2 R_X(f)$

Output is WSS with modified mean, ACF, and PSD

Example: Thermal Noise 1(3) – Physics

A resistor:



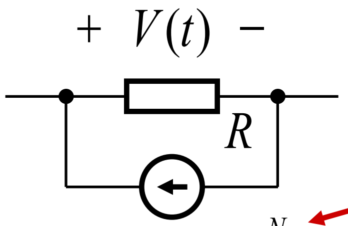
Thermal movements of electrons

$\Rightarrow$ Random local currents

$\Rightarrow$ Random local voltages

$\Rightarrow$ Random total voltage

Model:



$$V(t) = R \cdot I(t)$$

Enormous

$$I(t) = \sum_{k=1}^{N} I_k(t)$$

$\Rightarrow$ $I(t)$ Gaussian

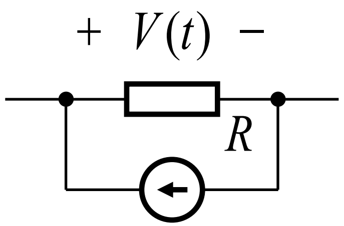
$\Rightarrow$ $V(t)$ Gaussian

Short pulses, almost unit impulses
$\Rightarrow I(t_1)$ & $I(t_2)$ almost independent for $t_1 \neq t_2$

White Gaussian Noise

Example: Thermal Noise 2(3) – PSD

Model cont'd:



White Gaussian noise $V(t)$ with

$$R_V(f) = \frac{N_0}{2} = 2kTR$$

N_0: Constant one-sided PSD

$k \approx 1.38 \cdot 10^{-23}$ J/K (Boltzmann's constant)

T = Absolute temperature in Kelvin.

R = Resistance Ω.

Power of noise:

$$E\{V^2(t)\} = r_V(0) = \int_{-\infty}^{\infty} R_V(f)df$$

Infinitely large!

What is wrong?

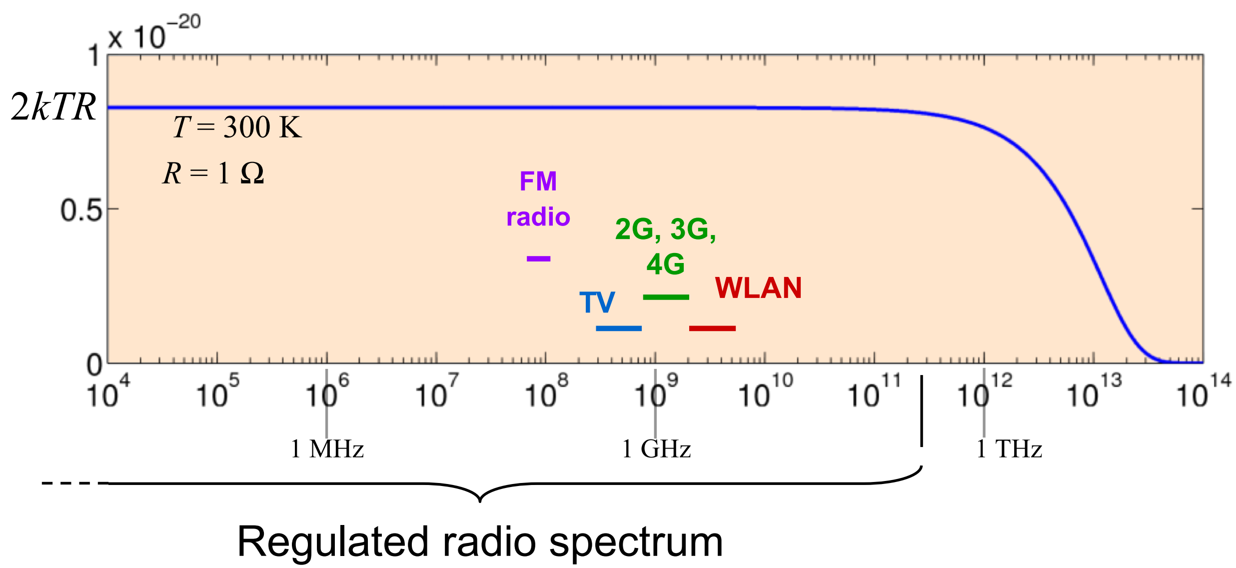
Example: Thermal Noise 3(3)
More Exact Model

Gaussian noise with

$$R_V(f) = \frac{2Rh|f|}{e^{h|f|/kT} - 1}$$

$h \approx 6.63 \cdot 10^{-34}$ Js
(Planck's constant)

Note: $R_V(f) \to 2kTR$ as $f \to 0$



White = Flat PSD in interval of operation

Flat PSD is fine!

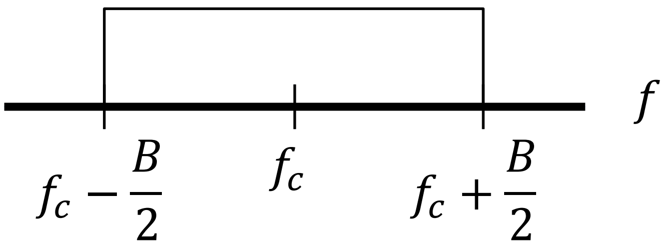
Example: Filtering of White Noise Process

White noise process
$X(t)$
$R_X(f) = N_0/2$

→ Ideal passband filter →

Filtered noise
$Y(t)$

- $H(f) = \begin{cases} 1, & \left||f| + f_c\right| < \frac{B}{2} \\ 0, & \text{elsewhere} \end{cases}$

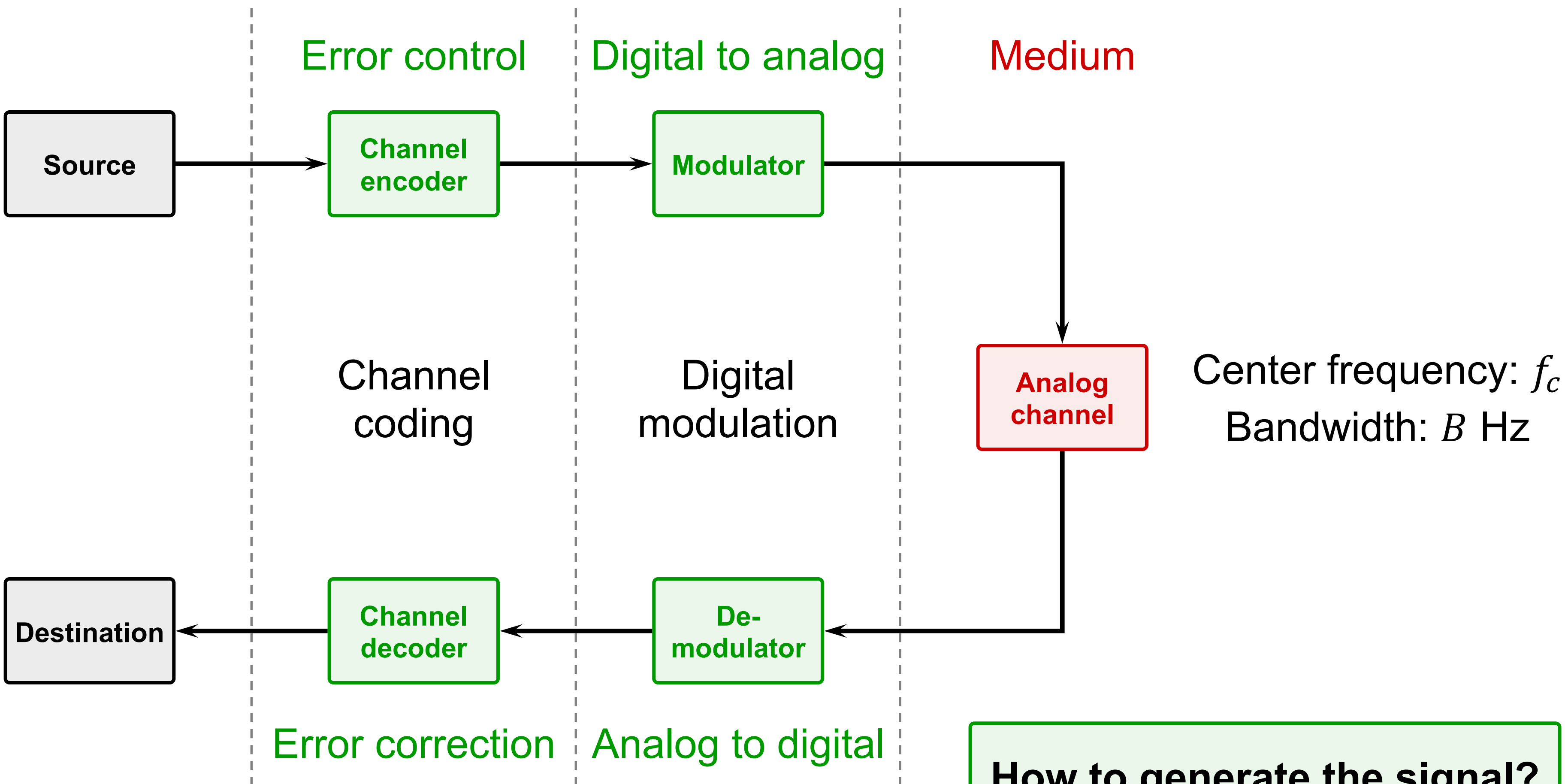


- What is the variance of the noise?

$$m_Y(t) = m_X H(0) = 0$$

$$\sigma_Y^2 = \mathrm{E}\{Y^2(t)\} = r_Y(0) = \int_{-\infty}^{\infty} R_Y(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = N_0 B$$

One-way Digital Communication System



How to generate the signal?

Pulse-Amplitude Modulation (Baseband)

Representation information:

- $s[n]$ = Time-discrete signal

Notation:

- T = Symbol interval
- $p(t)$ = Pulse-shape function

Pulse-Amplitude Modulation (PAM):

$$x(t) = \sum_{n} s[n]p(t - nT)$$

Nyquist Criterion for ISI-Free Communication

Sampling of $x(t)$ at time kT

$$z[k] = x(kT) = \sum_{n} s[n]p(kT - nT)$$

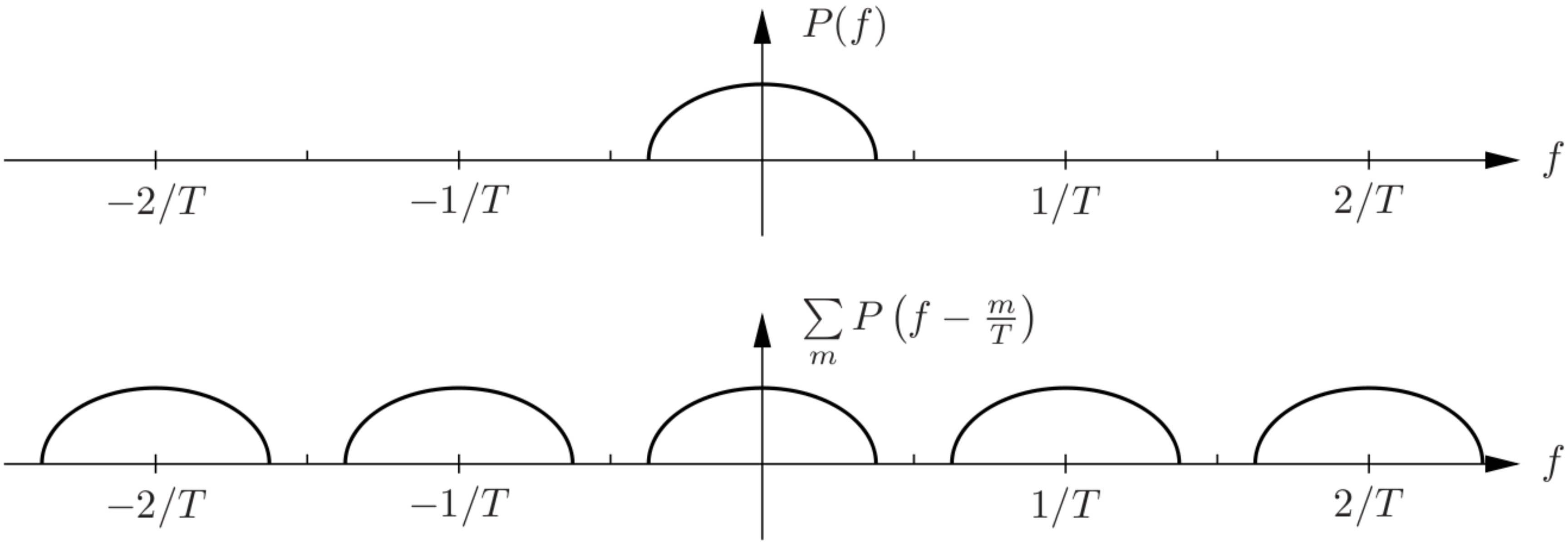
Inter-symbol interference (ISI): $z[k]$ depends on $s[n]$ for $k \neq n$.

Goal: ISI-free communication!

$z[k] = C \cdot s[k]$ for constant $C \neq 0$.

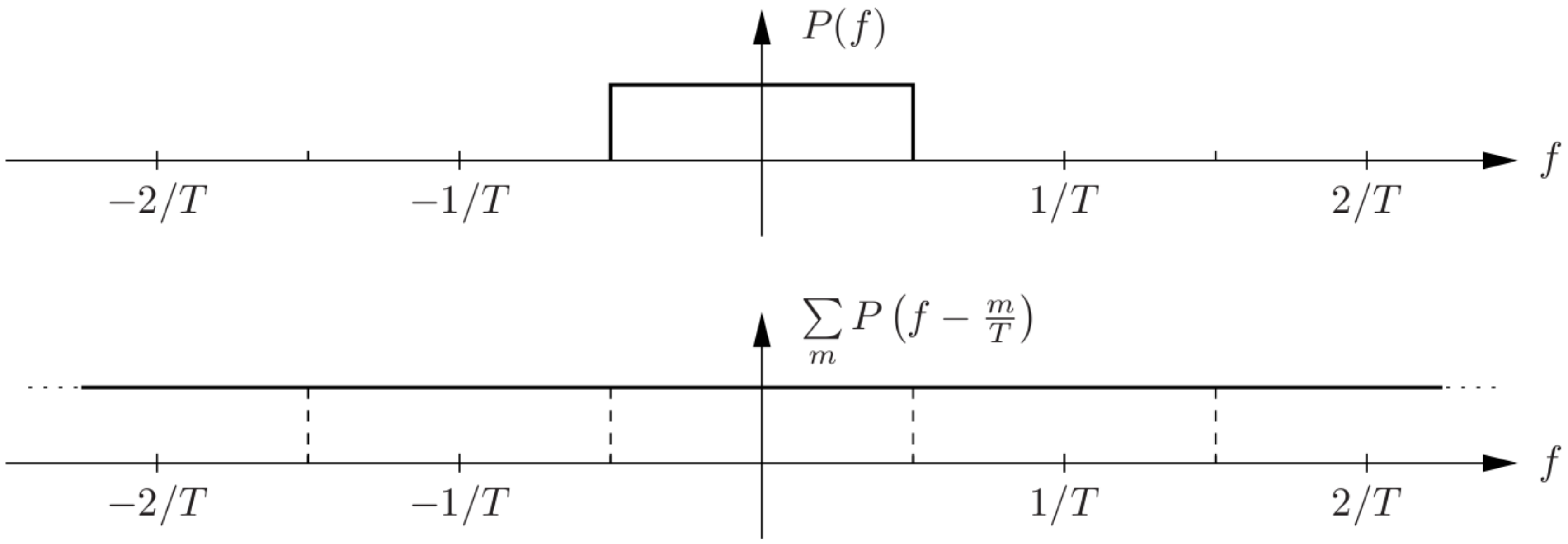
Achieved when $p(kT) = C\delta[k] \quad \Leftrightarrow \quad \underbrace{\frac{1}{T}\sum_{m=-\infty}^{\infty} P\left(f - \frac{m}{T}\right)}_{\text{Nyquist ISI criterion: Constant!}} = C$

A pulse with bandwidth less than $1/(2T)$ cannot be Nyquist



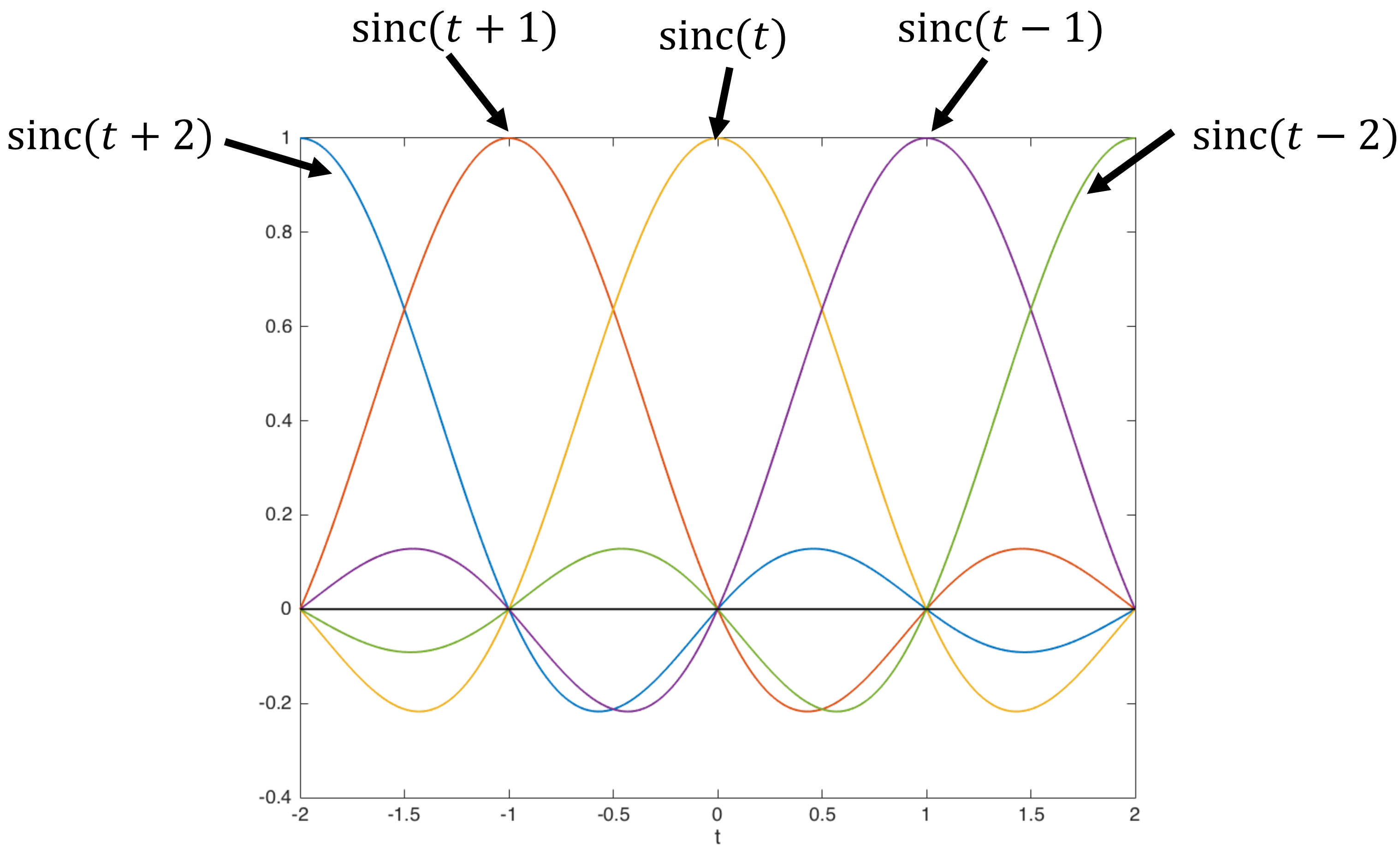
A pulse $p(t)$ is said to be **Nyquist** if $\sum_{m=-\infty}^{\infty} P\left(f - \frac{m}{T}\right)$ is constant

A pulse with bandwidth $1/(2T)$ has to be a sinc to be Nyquist



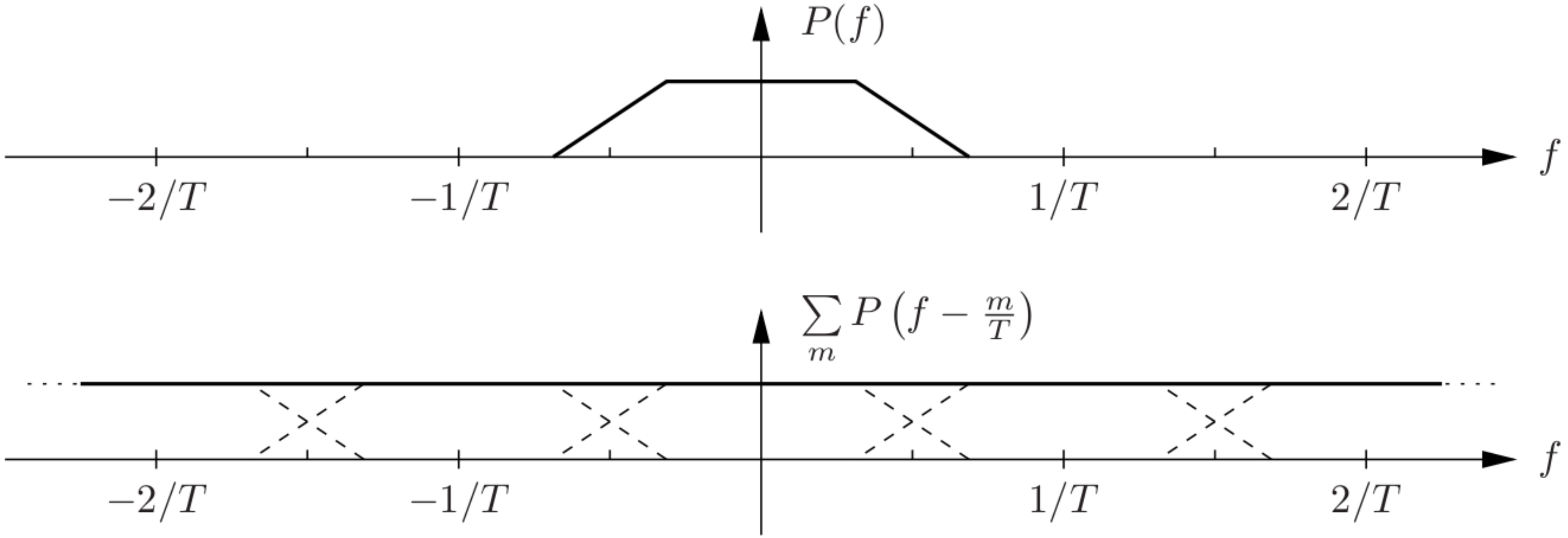
A pulse $p(t)$ is said to be **Nyquist** if $\sum_{m=-\infty}^{\infty} P\left(f - \frac{m}{T}\right)$ is constant

Interpretation of Nyquist in Time-domain



Problems: Non-causal, impossible to implement, sensitive to synchronization errors

A pulse with bandwidth more than $1/(2T)$ can have many shapes to be Nyquist



A pulse $p(t)$ is said to be **Nyquist** if $\sum_{m=-\infty}^{\infty} P\left(f - \frac{m}{T}\right)$ is constant

Nyquist Criterion with Receiver Filtering

Recall: Pulse-Amplitude Modulation (PAM):

$$x(t) = \sum_n s[n]p(t - nT)$$

Receiver filtering $\gamma(t), \Gamma(f)$:

$(p * \gamma)(t)$ acts as the pulse

$$(\gamma * x)(t) = \sum_n s[n](\gamma * p)(t - nT)$$

New Nyquist criterion:

$$\sum_{m=-\infty}^{\infty} \Gamma\left(f - \frac{m}{T}\right) P\left(f - \frac{m}{T}\right) = C$$

Called: $p(t)$ and $\gamma(t)$ are **Nyquist together**

Raised Cosine Pulse

A popular class of pulses bandwidth more than $1/(2T)$

- Parameter α determines excess bandwidth
  (Called: *roll-off factor* or *normalized excess bandwidth*)

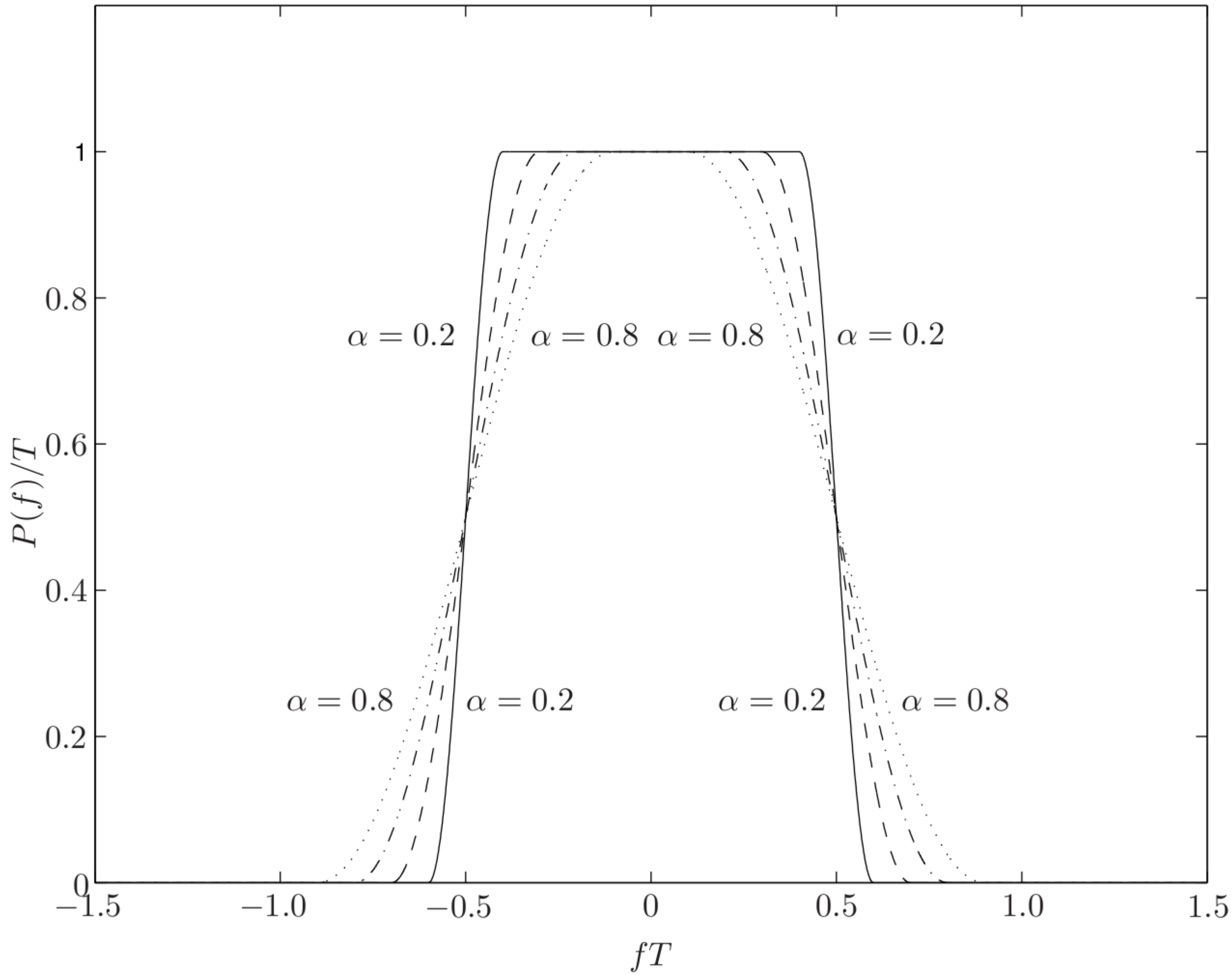
$$P(f) = \begin{cases} T, & |f| \leq \frac{1-\alpha}{2T} \\ \frac{T}{2}\left(1 + \cos\left(\frac{\pi T}{\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right)\right), & \frac{1-\alpha}{2T} < |f| < \frac{1+\alpha}{2T} \\ 0, & \text{elsewhere} \end{cases}$$

$$p(t) = \operatorname{sinc}\left(\frac{t}{T}\right)\frac{\cos(\alpha\pi t/T)}{1-(2\alpha t/T)^2}$$

Sinc: $p(t)$ decays as $1/|t|$,

Raised cosine: $p(t)$ decays as $1/|t|^3$,

Spectrum of a Raised Cosine Pulse



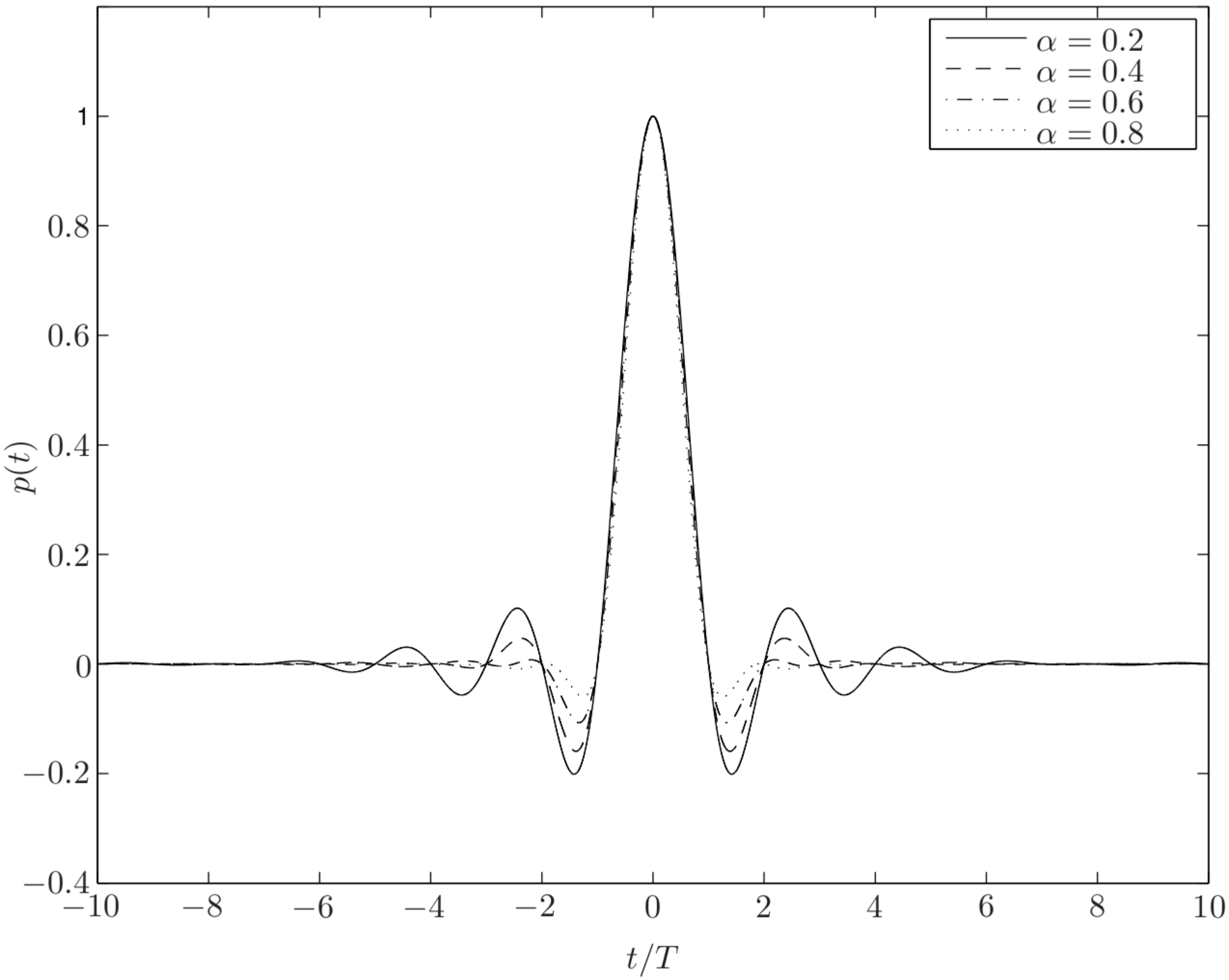
Bandwidth

$$B = \frac{1+\alpha}{2T}$$

Higher α:

More bandwidth

Impulse Response of a Raised Cosine Pulse



Higher α:

Decays faster

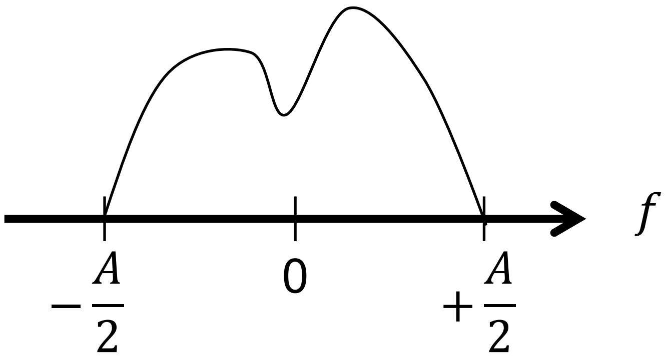
Root-Raised Cosine Pulse

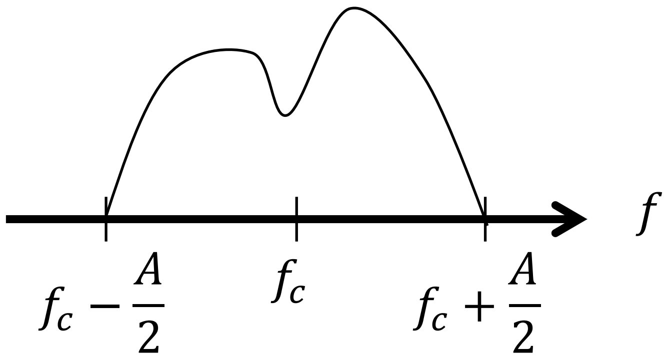
A pulse whose spectrum is the square root of a *Raised cosine* spectrum.

- Set $p(t)$ and $\gamma(t)$ as root-raised cosine pulses
- Nyquist together – form a raised cosine pulse together
- Very common in real communication systems!

Bandwidth in Baseband and Passband

Bandwidth: Distance from smallest to largest frequency





Bandwidth (baseband): $\frac{A}{2}$

Bandwidth (passband): A

Example: Raised Cosine Pulse (symbol time T)

Baseband:

$$B = \frac{1+\alpha}{2T}$$

Passband:

$$B = \frac{1+\alpha}{T}$$

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