TSDT14 Signal Theory

Solutions to the exam 2018-10-24

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These subtasks have to be treated correctly as partial fulfillment to pass the exam.

a. An ensemble average is the expectation of a function of samples of a stochastic process. Examples are the expectation of a process, its variance or its ACF.

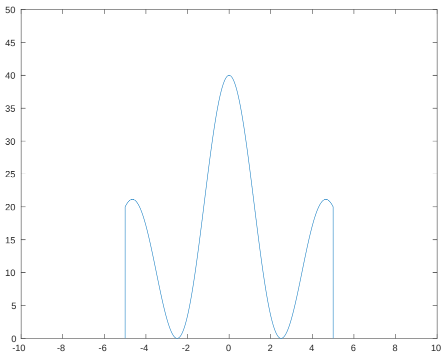
b. We have for the pulse-amplitude-modulated signal

$$R_Y(f) = \frac{1}{T}\left|P(f)\right|^2 R_X[fT]$$

Plugging in everything given in the task, we get

$$R_Y(f) = $$
$$= \begin{cases} 20\cos^2(2\pi f/10)\left(1+\cos(2\pi f/20)\right), & |f| < 5 \text{ Hz}, \\ 0, & \text{elsewhere} \end{cases}$$
$$= \begin{cases} 40\cos^2(2\pi f/10)\cos^2(2\pi f/40), & |f| < 5 \text{ Hz}, \\ 0, & \text{elsewhere} \end{cases}$$

Graphically:



c. We have a stochastic variable that is uniformly distributed on the interval $[-A, A]$. It is uniformly quantized with $N = 256$ quantization steps and saturation levels $\pm A$. For the quantization step Δ, we have

$$N\Delta = 2A.$$

From that we find the quantization step

$$\Delta = \frac{2A}{N}.$$

The quantization error Q has according to the Tables and Formulas booklet on page 16 the power

$$P_Q = \frac{\Delta^2}{12} = \frac{A^2}{3N^2},$$

and that is our distorsion.

2
The mean of the input is $m_X = 0$. The filter is LTI. Therefore, the mean of the output is $m_Y = m_X \cdot H(0) = 0$. The ACF of the input is

$$r_X(\tau) = 10^3 \cdot \text{sinc}(10^3\tau).$$

Tables and Formulas, p. 18, gives us the PSD

$$R_X(f) = \text{rect}\left(\frac{f}{10^3}\right) = \begin{cases} 1, & |f| < 500, \\ 0, & \text{elsewhere.} \end{cases}$$

The given frequency response of the filter is

$$H(f) = \begin{cases} 1 - \frac{|f|}{10^3}, & |f| < 10^3, \\ 0, & \text{elsewhere.} \end{cases}$$

Since the system is LTI, we have the output PSD

$$R_Y(f) = \left|H(f)\right|^2 R_X(f) = \begin{cases} \left(1 - \frac{|f|}{10^3}\right)^2, & |f| < 500, \\ 0, & \text{elsewhere.} \end{cases}$$

The mean is zero, which gives us

$$\sigma_Y^2 = \text{E}\{Y^2(t)\} = r_X(0) = \int_{-\infty}^{\infty} R_Y(f)\,df$$
$$= \int_{-500}^{500} \left(1 - \frac{|f|}{10^3}\right)^2 df = 2\int_{-500}^{0} \left(1 + \frac{f}{10^3}\right)^2 df.$$

The variable substitution $g = f + 10^3$ rewrites the expression above as

$$\sigma_Y^2 = 2 \int_{500}^{1000} \left(\frac{g}{10^3}\right)^2 df = \frac{1750}{3}$$

Finally, we calculate the probability

$$\Pr\{|Y| > 5\} = 2 \cdot Q\left(\frac{5}{\sigma_Y}\right) \approx 2 \cdot Q(0.21) \approx 0.83$$

3

We have two WSS signals $X(t)$ and $Y(t)$, both with mean zero. We also have a random phase Ψ that is uniformly distributed on $[0, 2\pi)$ and independent of both $X(t)$ and $Y(t)$. We are interested in the signal

$$Z(t) = X(t)\cos(2\pi f_c t + \Psi) + Y(t)\sin(2\pi f_c t + \Psi),$$

where f_c is a carrier frequency.

a. We are to prove that $Z(t)$ can be WSS even if $X(t)$ and $Y(t)$ are dependent, and determine what demands have to be fulfilled then.

First, we check the mean:

$$\begin{aligned} m_Z &= \mathrm{E}\{X(t)\cos(2\pi f_c t + \Psi) + Y(t)\sin(2\pi f_c t + \Psi)\} \\ &= m_X \mathrm{E}\{\cos(2\pi f_c t+\Psi)\} + m_Y \mathrm{E}\{\sin(2\pi f_c t+\Psi)\} \\ &= 0, \end{aligned}$$

which is constant. Here, we have used the independence of Ψ and each of the two signals $X(t)$ and $Y(t)$ in the second equality.

Next, we check the ACF:

$$\begin{aligned} &r_Z(t+\tau, t) = \\ &= \mathrm{E}\Big\{\Big(X(t+\tau)\cos\big(2\pi f_c(t+\tau)+\Psi\big) + \\ &\qquad + Y(t+\tau)\sin\big(2\pi f_c(t+\tau)+\Psi\big)\Big) \\ &\qquad \cdot \Big(X(t)\cos(2\pi f_c t+\Psi) + Y(t)\sin(2\pi f_c t+\Psi)\Big)\Big\} \\ &= \mathrm{E}\Big\{X(t+\tau)X(t)\Big\}\cdot\mathrm{E}\Big\{\cos\big(2\pi f_c(t+\tau)+\Psi\big)\cos(2\pi f_c t+\Psi)\Big\} \\ &\quad + \mathrm{E}\Big\{X(t+\tau)Y(t)\Big\}\cdot\mathrm{E}\Big\{\cos\big(2\pi f_c(t+\tau)+\Psi\big)\sin(2\pi f_c t+\Psi)\Big\} \\ &\quad + \mathrm{E}\Big\{Y(t+\tau)X(t)\Big\}\cdot\mathrm{E}\Big\{\sin\big(2\pi f_c(t+\tau)+\Psi\big)\cos(2\pi f_c t+\Psi)\Big\} \\ &\quad + \mathrm{E}\Big\{Y(t+\tau)Y(t)\Big\}\cdot\mathrm{E}\Big\{\sin\big(2\pi f_c(t+\tau)+\Psi\big)\sin(2\pi f_c t+\Psi)\Big\} \end{aligned}$$

Let us solve or identify each of the involved expectations separately. We start by identifying those four:

$$\begin{aligned} \mathrm{E}\{X(t+\tau)X(t)\} &= r_X(\tau), \\ \mathrm{E}\{X(t+\tau)Y(t)\} &= r_{X,Y}(t+\tau, t), \\ \mathrm{E}\{Y(t+\tau)X(t)\} &= r_{Y,X}(t+\tau, t), \\ \mathrm{E}\{Y(t+\tau)Y(t)\} &= r_Y(\tau). \end{aligned}$$

Then we solve the remaing four, containing trigonometric functions:

$$\begin{aligned} &\mathrm{E}\Big\{\cos\big(2\pi f_c(t+\tau)+\Psi\big)\cos(2\pi f_c t+\Psi)\Big\} = \\ &= \frac{1}{2}\mathrm{E}\Big\{\cos\big(2\pi f_c(2t+\tau)+2\Psi\big) + \cos(2\pi f_c \tau)\Big\} \\ &= \frac{1}{2}\cos(2\pi f_c \tau), \\ &\mathrm{E}\Big\{\cos\big(2\pi f_c(t+\tau)+\Psi\big)\sin(2\pi f_c t+\Psi)\Big\} = \\ &= \frac{1}{2}\mathrm{E}\Big\{\sin\big(2\pi f_c(2t+\tau)+2\Psi\big) - \sin(2\pi f_c \tau)\Big\} \\ &= -\frac{1}{2}\sin(2\pi f_c \tau), \\ &\mathrm{E}\Big\{\sin\big(2\pi f_c(t+\tau)+\Psi\big)\cos(2\pi f_c t+\Psi)\Big\} = \\ &= \frac{1}{2}\mathrm{E}\Big\{\sin\big(2\pi f_c(2t+\tau)+2\Psi\big) + \sin(2\pi f_c \tau)\Big\} \\ &= \frac{1}{2}\sin(2\pi f_c \tau), \\ &\mathrm{E}\Big\{\sin\big(2\pi f_c(t+\tau)+\Psi\big)\sin(2\pi f_c t+\Psi)\Big\} = \\ &= \frac{1}{2}\mathrm{E}\Big\{-\cos\big(2\pi f_c(2t+\tau)+2\Psi\big) + \cos(2\pi f_c \tau)\Big\} \\ &= \frac{1}{2}\cos(2\pi f_c \tau). \end{aligned}$$

In all four cases above, the expectations of the sines and cosines with 2Ψ in their arguments result in zero, since those become scaled integrals of two full periods of a sine. Totally, we have the ACF

$$\begin{aligned} &r_Z(t+\tau, t) = \\ &= \frac{1}{2}r_X(\tau)\cdot\cos(2\pi f_c\tau) - \frac{1}{2}r_{X,Y}(t+\tau,t)\cdot\sin(2\pi f_c\tau) \\ &\quad + \frac{1}{2}r_{Y,X}(t+\tau,t)\cdot\sin(2\pi f_c\tau) + \frac{1}{2}r_Y(\tau)\cdot\cos(2\pi f_c\tau) \\ &= \frac{1}{2}\big(r_X(\tau)+r_Y(\tau)\big)\cdot\cos(2\pi f_c\tau) \\ &\quad + \frac{1}{2}\big(r_{Y,X}(t+\tau,t) - r_{X,Y}(t+\tau,t)\big)\cdot\sin(2\pi f_c\tau). \end{aligned}$$

We notice that $r_Z(t+\tau, t)$ does not depend on t if

$$r_{Y,X}(t+\tau, t) - r_{X,Y}(t+\tau, t)$$

does not depend on t. That is therefore the criterion that has to hold.

b. A simple explicit example where $X(t)$ and $Y(t)$ are dependent, and where $Z(t)$ is WSS, is if the two processes are the same, i.e. if we have $X(t) = Y(t)$, and we have

$$\begin{aligned} Z(t) &= X(t)\big(\cos(2\pi f_c t + \Psi) + \sin(2\pi f_c t + \Psi)\big) \\ &= \sqrt{2}X(t)\cos(2\pi f_c t + \Psi - \pi/4), \end{aligned}$$

which is ordinary AM. Then we have

$$r_{Y,X}(t+\tau, t) - r_{X,Y}(t+\tau, t) = r_X(\tau) - r_X(\tau) = 0,$$

and our criterion from **a** is fulfilled.

A solution that does not use the result from **a** could be to observe that the additional phase-shift $-\pi/4$ does not change anything significant. The normal expression for AM holds, given on Page 14 in T&F. That expression does not depend on t. The mean is zero in a similar way as above in **a**.

Note: There do indeed exist dependent processes $X(t)$ and $Y(t)$ such that

$$r_{Y,X}(t+\tau, t) - r_{X,Y}(t+\tau, t)$$

depends on t. One such example is $Y(t) = X(t)\cos(2\pi f_1 t)$. We can also observe that if $X(t)$ and $Y(t)$ are jointly WSS, then both terms in

$$r_{Y,X}(t+\tau, t) - r_{X,Y}(t+\tau, t)$$

are independent of t, resulting in $Z(t)$ being WSS. Finally, if $X(t)$ and $Y(t)$ are independent processes, then we have

$$r_{Y,X}(t+\tau, t) = r_{X,Y}(t+\tau, t) = m_X m_Y.$$

That results in

$$r_{Y,X}(t+\tau, t) - r_{X,Y}(t+\tau, t) = 0.$$

4

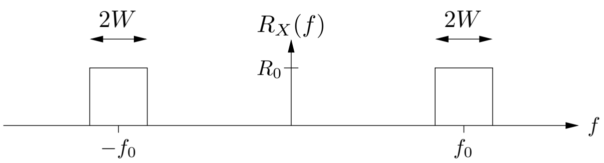
For the squarer we have

$$r_Y(\tau) = r_X^2(0) + 2r_X^2(\tau)$$

since $X(t)$ is Gaussian with mean 0 (T&F, p. 15). This gives us

$$R_Y(f) = r_X^2(0)\delta(f) + 2(R_X * R_X)(f)$$

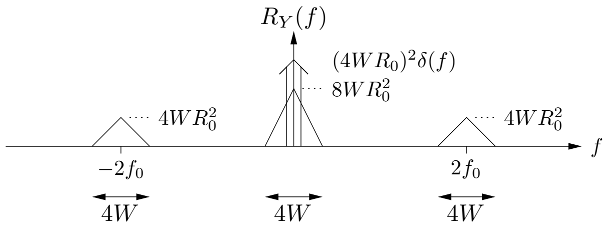
The PSD $R_X(f)$ graphically:



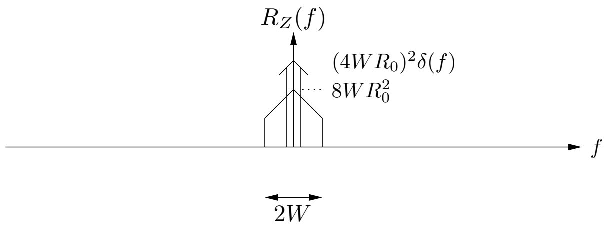
We have

$$r_X(0) = \int_{-\infty}^{\infty} R_X(f)\, df = 4WR_0.$$

The PSD $R_Y(f)$ graphically:



After LP-filtering, we have the following PSD:



Finally, we integrate and get the power

$$\begin{aligned} P_Z = E\{Z^2(t)\} &= \int_{-\infty}^{\infty} R_Z(f)\; df \\ &= 16W^2R_0^2 + 2W \cdot 4WR_0^2 + \frac{1}{2}2W \cdot 4WR_0^2 \\ &= 28W^2R_0^2. \end{aligned}$$

5

We are given a WSS time-discrete process $X[n]$, and the related process

$$Y[n] = \begin{cases} X[n+1], & n \text{ even}, \\ X[n-1], & n \text{ odd}. \end{cases}$$

a. We want to express $r_Y[n+k, n]$ i terms of $r_X[k]$. We have four cases, depending on if k and n are even or odd.

For the case where both k and n are even, we have

$$\begin{aligned} r_Y[n+k, n] &= \mathrm{E}\big\{Y[n+k]Y[n]\big\} = \\ &= \mathrm{E}\big\{X[n+k+1]X[n+1]\big\} = r_X[k]. \end{aligned}$$

For the case where k is even and n is odd, we have

$$r_Y[n+k,n] = \mathrm{E}\{Y[n+k]Y[n]\} = \\ = \mathrm{E}\{X[n+k-1]X[n-1]\} = r_X[k].$$

For the case where k is odd and n is even, we have

$$r_Y[n+k,n] = \mathrm{E}\{Y[n+k]Y[n]\} = \\ = \mathrm{E}\{X[n+k-1]X[n+1]\} = r_X[k-2].$$

For the case where both k and n are odd, we have

$$r_Y[n+k,n] = \mathrm{E}\{Y[n+k]Y[n]\} = \\ = \mathrm{E}\{X[n+k+1]X[n-1]\} = r_X[k+2].$$

b. First the mean: The mean m_X is constant since $X[n]$ is WSS. Obviously, $m_Y = m_X$, since reordering the samples means reordering the mean, and m_X is constant.

Then the ACF: According to part **a**, for even k there is no dependence on n, but for odd k there is in general a dependence on n. Thus, in general $Y[n]$ is not WSS. To remove the dependence on n, we must demand that

$$r_X[k-2] = r_X[k+2]$$

holds for all odd k.

6

We have a circular, WSS, complex process $X(t)$, and the related process $Y(t) = e^{j\phi}X(t)$ for a fixed real-valued ϕ.

a. We want to show that $X(t)$ and $Y(t)$ have the same PSD, which is equivalent to showing that $X(t)$ and $Y(t)$ have the same ACF, since the PSD of a process is the Fourier transform of the ACF of the same process. The ACF of the process $Y(t)$ is given by

$$\begin{aligned} r_Y(\tau) &= \mathrm{E}\{Y(t+\tau)Y^*(t)\} \\ &= \mathrm{E}\{e^{j\phi}X(t+\tau)e^{-j\phi}X^*(t)\} \\ &= \mathrm{E}\{X(t+\tau)X^*(t)\} \\ &= r_X(\tau). \end{aligned}$$

So, they do in fact have the same ACF, and thus also the same PSD.

b. Let $X_{\mathrm{I}}(t)$ and $X_{\mathrm{Q}}(t)$ be the real and imaginary parts, respectively, of the complex signal $X(t)$, i.e. we have

$$X(t) = X_{\mathrm{I}}(t) + jX_{\mathrm{Q}}(t).$$

$X(t)$ is WSS and circular. Thus, we have

$$r_{X_{\mathrm{I}}}(\tau) = r_{X_{\mathrm{Q}}}(\tau), \\ r_{X_{\mathrm{I}},X_{\mathrm{Q}}}(\tau) = -r_{X_{\mathrm{Q}},X_{\mathrm{I}}}(\tau).$$

According to the task formulation, we also have

$$\begin{aligned} Y(t) &= e^{j\phi}X(t) \\ &= \big(\cos(\phi) + j\sin(\phi)\big)\big(X_{\mathrm{I}}(t) + jX_{\mathrm{Q}}(t)\big). \end{aligned}$$

Similarily, let $Y_{\mathrm{I}}(t)$ and $Y_{\mathrm{Q}}(t)$ be the real and imaginary part, respectively, of $Y(t)$. Then we have

$$\begin{aligned} Y_{\mathrm{I}}(t) &= \mathrm{Re}\{Y(t)\} \\ &= \cos(\phi)X_{\mathrm{I}}(t) - \sin(\phi)X_{\mathrm{Q}}(t), \\ Y_{\mathrm{Q}}(t) &= \mathrm{Im}\{Y(t)\} \\ &= \cos(\phi)X_{\mathrm{Q}}(t) + \sin(\phi)X_{\mathrm{I}}(t). \end{aligned}$$

Our task is now to check if

$$r_{Y_{\mathrm{I}}}(\tau) = r_{Y_{\mathrm{Q}}}(\tau), \\ r_{Y_{\mathrm{I}},Y_{\mathrm{Q}}}(\tau) = -r_{Y_{\mathrm{Q}},Y_{\mathrm{I}}}(\tau),$$

hold for all real ϕ.

We have

$$\begin{aligned} & r_{Y_{\mathrm{I}}}(\tau) = \\ &= \mathrm{E}\Big\{\big(\cos(\phi)X_{\mathrm{I}}(t+\tau) - \sin(\phi)X_{\mathrm{Q}}(t+\tau)\big)\cdot \\ &\qquad \cdot\big(\cos(\phi)X_{\mathrm{I}}(t) - \sin(\phi)X_{\mathrm{Q}}(t)\big)\Big\} \\ &= \cos^2(\phi)r_{X_{\mathrm{I}}}(\tau) + \sin^2(\phi)r_{X_{\mathrm{Q}}}(\tau) \\ &\quad - \cos(\phi)\sin(\phi)\big(r_{X_{\mathrm{I}},X_{\mathrm{Q}}}(\tau) + r_{X_{\mathrm{Q}},X_{\mathrm{I}}}(\tau)\big) \\ &= \big(\cos^2(\phi) + \sin^2(\phi)\big)r_{X_{\mathrm{I}}}(\tau) \\ &\quad - \cos(\phi)\sin(\phi)\big(r_{X_{\mathrm{I}},X_{\mathrm{Q}}}(\tau) - r_{X_{\mathrm{I}},X_{\mathrm{Q}}}(\tau)\big) \\ &= r_{X_{\mathrm{I}}}(\tau), \end{aligned}$$

where we have use the assumption that $X(t)$ is circular in the second last equality. This holds regardless of ϕ. Therefore, we also have

$$r_{Y_{\mathrm{Q}}}(\tau) = r_{X_{\mathrm{I}}}(\tau).$$

We have thus showed the equality

$$r_{Y_{\mathrm{I}}}(\tau) = r_{Y_{\mathrm{Q}}}(\tau).$$

Then we have

$$
\begin{aligned}
&r_{Y_\mathrm{I},Y_\mathrm{Q}}(\tau) = \\
&= \mathrm{E}\Big\{\big(\cos(\phi)X_\mathrm{I}(t+\tau) - \sin(\phi)X_\mathrm{Q}(t+\tau)\big)\cdot \\
&\qquad \cdot \big(\cos(\phi)X_\mathrm{Q}(t) + \sin(\phi)X_\mathrm{I}(t)\big)\Big\} \\
&= \cos^2(\phi) r_{X_\mathrm{I},X_\mathrm{Q}}(\tau) - \sin^2(\phi) r_{X_\mathrm{Q},X_\mathrm{I}}(\tau) \\
&\quad + \cos(\phi)\sin(\phi)\big(r_{X_\mathrm{I}}(\tau) - r_{X_\mathrm{Q}}(\tau)\big) \\
&= \big(\cos^2(\phi) + \sin^2(\phi)\big) r_{X_\mathrm{I},X_\mathrm{Q}}(\tau) \\
&\quad + \cos(\phi)\sin(\phi)\big(r_{X_\mathrm{I}}(\tau) - r_{X_\mathrm{I}}(\tau)\big) \\
&= r_{X_\mathrm{I},X_\mathrm{Q}}(\tau),
\end{aligned}
$$

where we again have use the assumption that $X(t)$ is circular in the second last equality. Similarily, we have

$$r_{Y_\mathrm{Q},Y_\mathrm{I}}(\tau) = -r_{X_\mathrm{I},X_\mathrm{Q}}(\tau).$$

We have thus showed the equality

$$r_{Y_\mathrm{I},Y_\mathrm{Q}}(\tau) = -r_{Y_\mathrm{Q},Y_\mathrm{I}}(\tau).$$

To conclude, $Y(t)$ is circular for all ϕ.