TSDT14 Signal Theory

Lecture 11

Multi-Dimensional Processes – Primarily 2-D

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Example: 2-D Filtering

Mr.
Angry

Mrs.
Calm

Source: http://www.grand-illusions.com/opticalillusions/angry_and_calm/
Does not seem to be there anymore.

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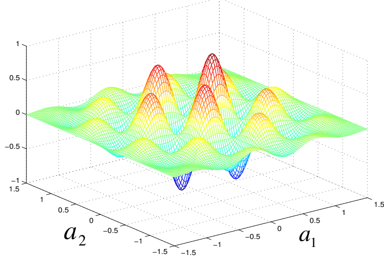
Multi-Dimensional Signals & Systems

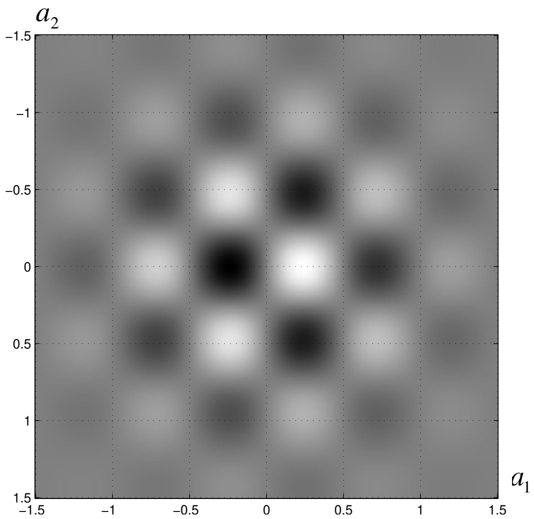
Two-Dimensional Signals

A function of two variables

$x(a_1, a_2)$ (space-continuous)

$x[n_1, n_2]$ (space-discrete)





Separable signals:

$$x(a_1, a_2) = x_1(a_1) \cdot x_2(a_2)$$

$$x[n_1, n_2] = x_1[n_1] \cdot x_2[n_2]$$

Two-Dimensional Stochastic Processes

Mean: $m_X(a_1, a_2) = \mathrm{E}\{X(a_1, a_2)\}$

Auto-correlation: $r_X(a_1, a_2; a_1 + b_1, a_2 + b_2) = \mathrm{E}\{X(a_1, a_2) X(a_1 + b_1, a_2 + b_2)\}$

Wide-sense stationarity:

Both $m_X(a_1, a_2)$ and $r_X(a_1, a_2; a_1 + b_1, a_2 + b_2)$ independent of (a_1, a_2).

Simplified notation: m_X and $r_X(b_1, b_2)$

Classification of Systems

Linearity:

Input $x_1(a_1, a_2)$ gives output $y_1(a_1, a_2)$.
Input $x_2(a_1, a_2)$ gives output $y_2(a_1, a_2)$.
Then input $a \cdot x_1(a_1, a_2) + b \cdot x_2(a_1, a_2)$
gives output $a \cdot y_1(a_1, a_2) + b \cdot y_2(a_1, a_2)$.

Space-invariance:

Input $x(a_1, a_2)$ gives output $y(a_1, a_2)$.
Then input $x(a_1 - b_1, a_2 - b_2)$
gives output $y(a_1 - b_1, a_2 - b_2)$.

LSI: Both linear and space-invariant.

Similarily for space-discrete systems

Two-Dimensional Convolution

For LSI systems, the output is given by a two-dimensional convolution of the input and the impulse response of the filter.

Definition:

$$(x \circledast h)(a_1, a_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(b_1, b_2) h(a_1 - b_1, a_2 - b_2)\, db_1 db_2$$

Separable signals:

$$(x \circledast h)(a_1, a_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(b_1) x_2(b_2)\, h_1(a_1 - b_1)\, h_2(a_2 - b_2)\, db_1 db_2$$

$$= \int_{-\infty}^{\infty} x_1(b_1) h_1(a_1 - b_1)\, db_1 \int_{-\infty}^{\infty} x_2(b_2) h_2(a_2 - b_2)\, db_2$$

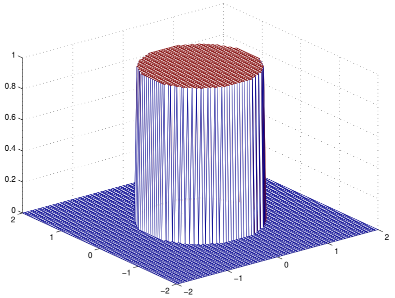
$$= (x_1 * h_1)(a_1) \cdot (x_2 * h_2)(a_2)$$

Space-discrete:

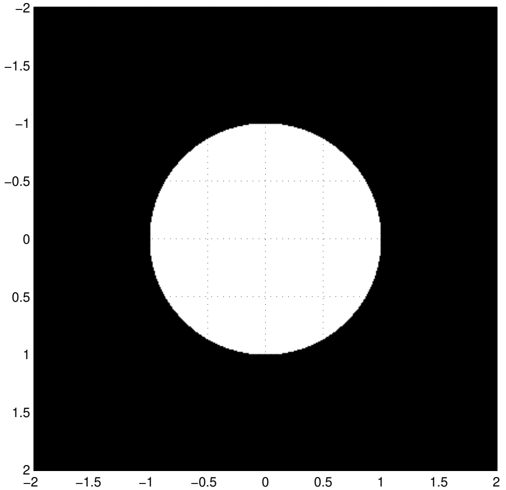
$$(x \circledast h)[n_1, n_2] = \sum_{k_1} \sum_{k_2} x[k_1, k_2] \cdot h[n_1 - k_1, n_2 - k_2]$$

Non-Separable Convolution 1(2)

$$x(a_1,a_2)=h(a_1,a_2)=\begin{cases}1, & a_1^2+a_2^2<1\\ 0, & \text{elsewhere}\end{cases}$$



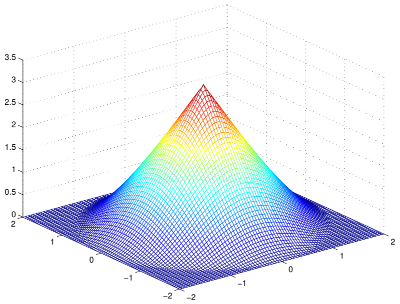
Non-separable signals

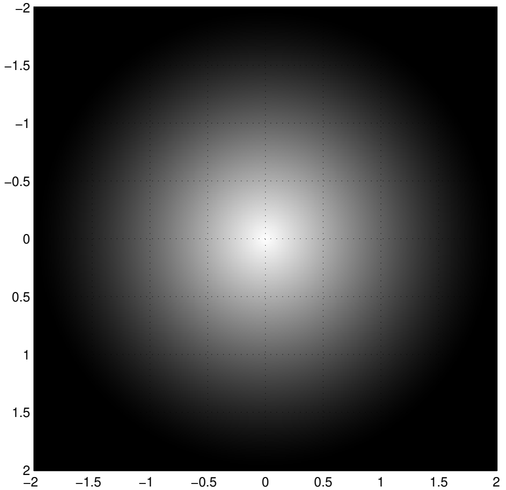


Non-Separable Convolution 2(2)

$$y(a_1,a_2)=(x\circledast h)(a_1,a_2)$$

$$=\begin{cases}2\arccos(a/2)-a\sqrt{1-(a/2)^2}, & a^2=a_1^2+a_2^2<4\\ 0, & \text{elsewhere}\end{cases}$$





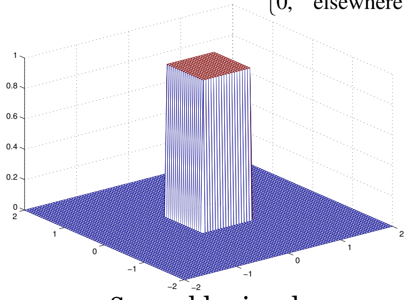
Separable Convolution 1(3)

$$x(a_1,a_2)=h(a_1,a_2)=$$

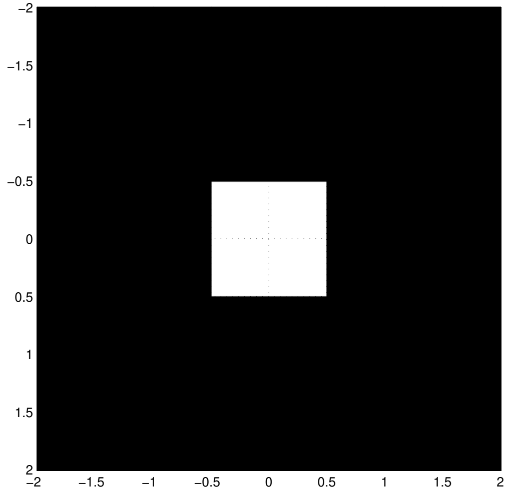
$$=\begin{cases}1, & |a_1|<1/2 \quad \text{and} \quad |a_2|<1/2\\ 0, & \text{elsewhere}\end{cases}$$

$$=z(a_1)\cdot z(a_2)$$

with $z(a)=\mathrm{rect}(a)=\begin{cases}1, & |a|<1/2\\ 0, & \text{elsewhere}\end{cases}$



Separable signals

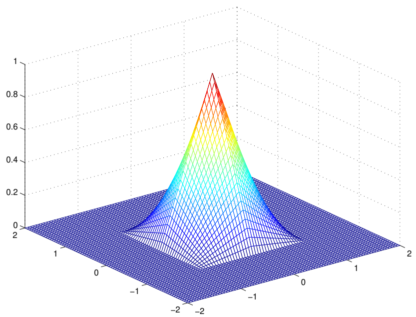


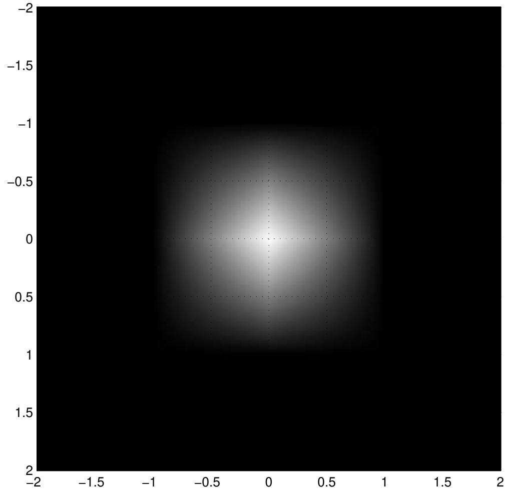
Separable Convolution 2(3)

$$y(a_1,a_2)=(x\circledast h)(a_1,a_2)$$

$$=\max\{0,1-|a_1|\}\cdot\max\{0,1-|a_2|\}$$

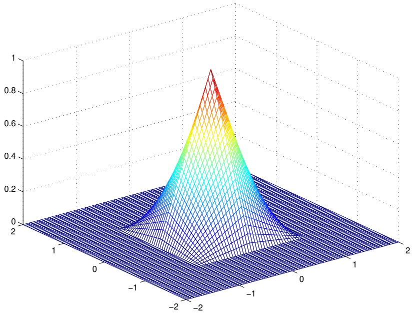
$$=\mathrm{triangle}(a_1)\cdot\mathrm{triangle}(a_2)$$

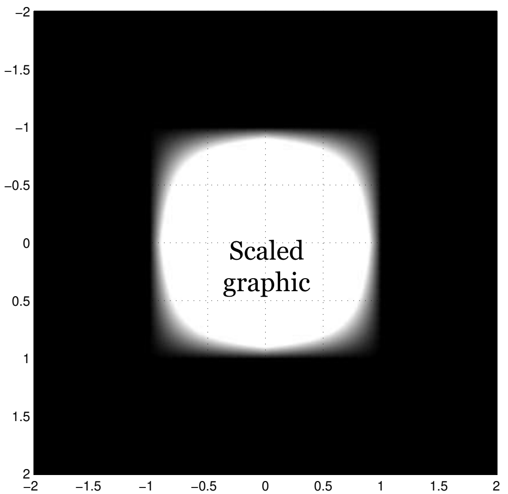




Separable Convolution 3(3)

$$y(a_1,a_2)=(x\circledast h)(a_1,a_2)$$
$$=\max\{0,1-|a_1|\}\cdot\max\{0,1-|a_2|\}$$
$$=\mathrm{triangle}(a_1)\cdot\mathrm{triangle}(a_2)$$





2-D Space-Continuous Fourier Transform

Definition: $$X(f_1,f_2)=\mathcal{F}\{x(a_1,a_2)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(a_1,a_2)e^{-j2\pi(f_1a_1+f_2a_2)}da_1da_2$$

Inverse: $$x(a_1,a_2)=\mathcal{F}^{-1}\{X(f_1,f_2)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X(f_1,f_2)e^{j2\pi(f_1a_1+f_2a_2)}df_1df_2$$

Properties: $$\mathcal{F}\{x(a_1,a_2)\}=\mathcal{F}_1\{\mathcal{F}_2\{x(a_1,a_2)\}\}=\mathcal{F}_2\{\mathcal{F}_1\{x(a_1,a_2)\}\}$$
$$\mathcal{F}\{x_1(a_1)\cdot x_2(a_2)\}=\mathcal{F}_1\{x_1(a_1)\}\cdot\mathcal{F}_2\{x_2(a_2)\}$$
$$\mathcal{F}\{(x\circledast h)(a_1,a_2)\}=X(f_1,f_2)\cdot H(f_1,f_2)$$
$$\mathcal{F}\{x(a_1,a_2)\cdot h(a_1,a_2)\}=(X\circledast H)(f_1,f_2)$$

2-D Space-Discrete Fourier Transform

Definition: $$X[\theta_1,\theta_2]=\mathcal{F}\{x[n_1,n_2]\}=\sum_{n_1}\sum_{n_2}x[n_1,n_2]e^{-j2\pi(\theta_1n_1+\theta_2n_2)}$$

Inverse: $$x[a_1,a_2]=\mathcal{F}^{-1}\{X[\theta_1,\theta_2]\}=\int_0^1\int_0^1 X[\theta_1,\theta_2]e^{j2\pi(\theta_1n_1+\theta_2n_2)}d\theta_1d\theta_2$$

Properties: $X[\theta_1,\theta_2]=X[\theta_1+m_1,\theta_2+m_2]$ for integers m_1 and m_2.
$$\mathcal{F}\{x[n_1,n_2]\}=\mathcal{F}_1\{\mathcal{F}_2\{x[n_1,n_2]\}\}=\mathcal{F}_2\{\mathcal{F}_1\{x[n_1,n_2]\}\}$$
$$\mathcal{F}\{x_1[n_1]\cdot x_2[n_2]\}=\mathcal{F}_1\{x_1[n_1]\}\cdot\mathcal{F}_2\{x_2[n_2]\}$$
$$\mathcal{F}\{(x\circledast h)[n_1,n_2]\}=X[\theta_1,\theta_2]\cdot H[\theta_1,\theta_2]$$
$$\mathcal{F}\{x[n_1,n_2]\cdot h[n_1,n_2]\}=(X\circledast H)[\theta_1,\theta_2]\quad\text{(periodic conv)}$$

Power and Power Spectral Density

PSD: $$R_X(f_1,f_2)=\mathcal{F}\{r_X(b_1,b_2)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}r_X(b_1,b_2)e^{-j2\pi(f_1b_1+f_2b_2)}db_1db_2$$

Power: $$P_X=E\{X^2(a_1,a_2)\}=r_X(0,0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R_X(f_1,f_2)df_1df_2$$

Output of LSI-system if input is WSS:

Output: $$Y(a_1,a_2)=(X\circledast h)(a_1,a_2)$$

Mean: $$m_Y=m_X\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(a_1,a_2)\,da_1da_2=m_X\cdot H(0,0)$$

ACF: $$r_Y(b_1,b_2)=(h\circledast\tilde{h}\circledast r_X)(b_1,b_2)\quad\text{with}\quad\tilde{h}(a_1,a_2)=h(-a_1,-a_2)$$

PSD: $$R_Y(f_1,f_2)=|H(f_1,f_2)|^2R_X(f_1,f_2)$$

Two-Dimensional Sampling

Deterministic (for rectangular grid):

Sampling: $y[n_1,n_2]=x(n_1A_1,n_2A_2)$

Sampling periods: A_1 and A_2

Spectrum: $Y[\theta_1,\theta_2]=\dfrac{1}{A_1A_2}\sum_{k_1}\sum_{k_2}X\left(\dfrac{\theta_1-k_1}{A_1},\dfrac{\theta_2-k_2}{A_2}\right)$

Probabilistic (for rectangular grid):

Sampling: $Y[n_1,n_2]=X(n_1A_1,n_2A_2)$

PSD: $R_Y[\theta_1,\theta_2]=\dfrac{1}{A_1A_2}\sum_{k_1}\sum_{k_2}R_X\left(\dfrac{\theta_1-k_1}{A_1},\dfrac{\theta_2-k_2}{A_2}\right)$

Two-Dimensional PAM

Deterministic (for rectangular grid):

PAM: $z(a_1,a_2)=\sum_{n_1}\sum_{n_2}y[n_1,n_2]p(a_1-n_1A_1,a_2-n_2A_2)$

Spectrum: $Z(f_1,f_2)=P(f_1,f_2)\cdot Y[f_1A_1,f_2A_2]$

Probabilistic (for rectangular grid):

PAM: $Z(a_1,a_2)=\sum_{n_1}\sum_{n_2}Y[n_1,n_2]p(a_1-n_1A_1-\Psi_1,a_2-n_2A_2-\Psi_2)$

Ψ_1 uniform on $[0,A_1)$ and Ψ_2 uniform on $[0,A_2)$

both independent of $Y[n_1,n_2]$ and of each other.

PSD: $R_Z(f_1,f_2)=\dfrac{1}{A_1A_2}|P(f_1,f_2)|^2R_Y[f_1A_1,f_2A_2]$

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