TSDT14 Signal Theory

Lecture 8
Analog Modulation – Part 2
Repetition of Sampling and PAM (deterministically)
Sampling and PAM of Stochastic Processes

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Broadband Angle Modulation

Modulation of a Gaussian process

Assumptions: $\Phi_\mathrm{d}(t)$ is an ergodic Gaussian process, mean 0.

Ψ is uniform on $[0,2\pi)$ and indep of $\Phi_\mathrm{d}(t)$.

Signal: $X(t) = A\cos\left(2\pi f_c t + \Phi_\mathrm{d}(t) + \Psi\right) \quad \Rightarrow \quad m_X = 0$

Phase dev.: $\Phi_\mathrm{d}(t)$

Freq. dev.: $F_\mathrm{d}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt}\Phi_\mathrm{d}(t)$

Relation: $R_{F_\mathrm{d}}(f) = f^2 R_{\Phi_\mathrm{d}}(f). \quad \Rightarrow \quad r_{F_\mathrm{d}}(\tau) = -\frac{1}{(2\pi)^2} \cdot \frac{d^2}{d\tau^2} r_{\Phi_\mathrm{d}}(\tau)$

Definitions: $\phi_\mathrm{d}^2 = 2r_{\Phi_\mathrm{d}}(0) \qquad f_\mathrm{d}^2 = 2r_{F_\mathrm{d}}(0),$

ACF of Angle Modulation 1(3)

$$\begin{aligned}
r_X(t+\tau,t) &= \mathrm{E}\left\{A\cos\Big(2\pi f_\mathrm{c}(t+\tau) + \Phi_\mathrm{d}(t+\tau) + \Psi\Big) \cdot A\cos\Big(2\pi f_\mathrm{c} t + \Phi_\mathrm{d}(t) + \Psi\Big)\right\} \\
&= \frac{A^2}{2}\mathrm{E}\left\{\cos\Big(2\pi f_\mathrm{c}(2t+\tau) + \Phi_\mathrm{d}(t+\tau) + \Phi_\mathrm{d}(t) + 2\Psi\Big)\right\} \\
&\qquad + \frac{A^2}{2}\mathrm{E}\left\{\cos\Big(2\pi f_\mathrm{c}\tau + \Phi_\mathrm{d}(t+\tau) - \Phi_\mathrm{d}(t)\Big)\right\}, \\
&= \frac{A^2}{2}\mathrm{E}\left\{\cos\Big(2\pi f_\mathrm{c}\tau + \Phi_\mathrm{d}(t+\tau) - \Phi_\mathrm{d}(t)\Big)\right\} \\
&= \frac{A^2}{2}\mathrm{E}\Big\{\mathrm{Re}\left\{e^{j[2\pi f_\mathrm{c}\tau + \Phi_\mathrm{d}(t+\tau) - \Phi_\mathrm{d}(t)]}\right\}\Big\} \\
&= \frac{A^2}{2}\mathrm{Re}\Big\{\mathrm{E}\left\{e^{j[\Phi_\mathrm{d}(t+\tau) - \Phi_\mathrm{d}(t)]}\right\} e^{j2\pi f_\mathrm{c}\tau}\Big\},
\end{aligned}$$

ACF of Angle Modulation 2(3)

We had: $r_X(t+\tau,t) = \frac{A^2}{2}\mathrm{Re}\Big\{\mathrm{E}\left\{e^{j[\Phi_\mathrm{d}(t+\tau) - \Phi_\mathrm{d}(t)]}\right\} e^{j2\pi f_\mathrm{c}\tau}\Big\}$

Temporary Gaussian variable: $Y = \Phi_\mathrm{d}(t+\tau) - \Phi_\mathrm{d}(t)$ (mean 0)

Then: $\mathrm{E}\left\{e^{jY}\right\} = \mathrm{E}\{\cos(Y)\} + j\mathrm{E}\{\sin(Y)\} = \mathrm{E}\{\cos(Y)\},$

Important observation: This is real valued.

Rewrite: $r_X(t+\tau,t) = \frac{A^2}{2}\mathrm{E}\left\{e^{jY}\right\}\mathrm{Re}\left\{e^{j2\pi f_\mathrm{c}\tau}\right\} = \frac{A^2}{2}\mathrm{E}\left\{e^{jY}\right\}\cos(2\pi f_\mathrm{c}\tau)$

We have, with $f = -1/2\pi$, using the identity $\mathcal{F}_x\{e^{-\pi(\breve{x}/T)^2}\} = T\,e^{-\pi(fT)^2}$:

$$\begin{aligned}
\mathrm{E}\left\{e^{jY}\right\} &= \mathrm{E}\left\{e^{-j2\pi fY}\right\} = \mathcal{F}_y\{f_Y(y)\} = \mathcal{F}_y\left\{\frac{1}{\sqrt{2\pi}\sigma_Y}e^{-y^2/2\sigma_Y^2}\right\} \\
&= e^{-(2\pi f\sigma_Y)^2/2} = e^{-\sigma_Y^2/2}
\end{aligned}$$

ACF of Angle Modulation 3(3)

We had: $r_X(t+\tau,t) = \frac{A^2}{2}\, e^{-\sigma_Y^2/2}\, \cos(2\pi f_c \tau)$

Recall: $Y = \Phi_{\mathrm{d}}(t+\tau) - \Phi_{\mathrm{d}}(t)$ (mean 0)

We have: $\sigma_Y^2 = \mathrm{E}\left\{\left(\Phi_{\mathrm{d}}(t+\tau) - \Phi_{\mathrm{d}}(t)\right)^2\right\} = 2r_{\Phi_{\mathrm{d}}}(0) - 2r_{\Phi_{\mathrm{d}}}(\tau).$

Mean and ACF independent of time *t*. Wide sense stationary!

$$m_X = 0 \qquad r_X(\tau) = \frac{A^2}{2}\cdot r_{\tilde{X}}(\tau)\cdot\cos(2\pi f_c\tau) \qquad r_{\tilde{X}}(\tau) = e^{r_{\Phi_{\mathrm{d}}}(\tau) - r_{\Phi_{\mathrm{d}}}(0)}$$

The introduced process $\tilde{X}(t)$ is a normalized baseband representation of $X(t)$.

PSD of Angle Modulation

We had: $r_X(\tau) = \frac{A^2}{2}\cdot r_{\tilde{X}}(\tau)\cdot\cos(2\pi f_c\tau)$ $\qquad r_{\tilde{X}}(\tau) = e^{r_{\Phi_{\mathrm{d}}}(\tau) - r_{\Phi_{\mathrm{d}}}(0)}$

Recall: $\phi_{\mathrm{d}}^2 = 2r_{\Phi_{\mathrm{d}}}(0)$ $\qquad f_{\mathrm{d}}^2 = 2r_{F_{\mathrm{d}}}(0),$

This holds if $r_{\Phi_{\mathrm{d}}}(\tau)$ has no periodic components and $\Phi_{\mathrm{d}}(t)$ has mean 0:

$$r_{\tilde{X}}(\tau) \approx m_{\tilde{X}}^2 + (1 - m_{\tilde{X}}^2)\cdot e^{-2\pi^2 r_{F_{\mathrm{d}}}(0)\tau^2} = m_{\tilde{X}}^2 + (1 - m_{\tilde{X}}^2)\cdot e^{-(\pi f_{\mathrm{d}}\tau)^2}$$

$$m_{\tilde{X}}^2 = e^{-r_{\Phi_{\mathrm{d}}}(0)} = e^{-\phi_{\mathrm{d}}^2/2}$$

Spectrum: $$R_{\tilde{X}}(f) \approx \underset{\underset{\text{Carrier}}{\uparrow}}{m_{\tilde{X}}^2\delta(f)} + \underset{\underset{\text{Sidebands}}{\uparrow}}{\frac{1 - m_{\tilde{X}}^2}{\sqrt{\pi}\cdot f_{\mathrm{d}}}\cdot e^{-(f/f_{\mathrm{d}})^2}}.$$

The only relation to $R_{\Phi_{\mathrm{d}}}(f)$ is through $r_{F_{\mathrm{d}}}(0)$ and $r_{\Phi_{\mathrm{d}}}(0)$.

Bandwidth of Angle Modulation

Define bandwidth B_X as the width of the frequency band, symmetrically around the carrier frequency that contains 90% of the power.

$$\int_{-B_X/2}^{B_X/2} R_{\tilde{X}}(f)\, df = 0.9.$$

Broadband $\Rightarrow$ $m_{\tilde{X}}^2$ very small. Ignore!

$$\int_{-B_X/2}^{B_X/2} \frac{1}{\sqrt{\pi}\cdot f_{\mathrm{d}}}\cdot e^{-(f/f_{\mathrm{d}})^2}\, df \approx 0.9.$$

We get:

$$\int_{B_X/2}^{\infty} \frac{1}{\sqrt{\pi}\cdot f_{\mathrm{d}}}\cdot e^{-(f/f_{\mathrm{d}})^2}\, df = Q\left(\frac{B_X/2}{f_{\mathrm{d}}/\sqrt{2}}\right) \approx 0.05. \qquad \Rightarrow \qquad B_X \approx 2.33 f_{\mathrm{d}}.$$

Compare Carson’s rule: Bandwidth is slightly more than $2{\cdot}f_{\mathrm{d,max}}$.

Special Cases: Broadband PM and FM

Phase modulation:

$$\Phi_{\mathrm{d}}(t) = a\, M(t),$$
$$r_{\Phi_{\mathrm{d}}}(\tau) = a^2\, r_M(\tau) = \frac{\phi_{\mathrm{d}}^2}{2}\cdot\frac{r_M(\tau)}{r_M(0)},$$
$$R_{\Phi_{\mathrm{d}}}(f) = a^2\, R_M(f) = \frac{\phi_{\mathrm{d}}^2}{2}\cdot\frac{R_M(f)}{r_M(0)}.$$

Frequency modulation

$$\Phi_{\mathrm{d}}(t) = a\int M(t)\, dt,$$
$$F_{\mathrm{d}}(t) = \frac{1}{2\pi}\cdot\frac{d}{dt}\Phi(t) = \frac{a}{2\pi}\, M(t),$$
$$r_{F_{\mathrm{d}}}(\tau) = \left(\frac{a}{2\pi}\right)^2 r_M(\tau) = \frac{f_{\mathrm{d}}^2}{2}\cdot\frac{r_M(\tau)}{r_M(0)},$$
$$R_{F_{\mathrm{d}}}(f) = \left(\frac{a}{2\pi}\right)^2 R_M(f) = \frac{f_{\mathrm{d}}^2}{2}\cdot\frac{R_M(f)}{r_M(0)}.$$

Narrowband Angle Modulation

We have:

$$X(t) = A\cos\left(2\pi f_c t + \Phi_d(t) + \Psi\right)$$
$$= A\cos\left(\Phi_d(t)\right)\cos\left(2\pi f_c t + \Psi\right) - A\sin\left(\Phi_d(t)\right)\sin\left(2\pi f_c t + \Psi\right)$$

Narrowband modulation, i.e. $\Phi_d(t)$ is small:

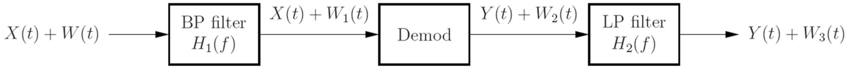
$$X(t) \approx A\cos(2\pi f_c t + \Psi) - A\,\Phi_d(t)\sin(2\pi f_c t + \Psi)\,.$$

Spectrum:

$$R_X(f) = \underbrace{\frac{A^2}{4}\left(\delta(f+f_c) + \delta(f-f_c)\right)}_{\text{Carrier}} + \underbrace{\frac{A^2}{4}\left(R_{\Phi_d}(f+f_c) + R_{\Phi_d}(f-f_c)\right)}_{\text{Sidebands}}$$

We get AM in the quadrature component.

AWGN Impact on Angle Modulation



Noise PSD: $R_W(f) = R_0$

Filter: $H_1(f) = \begin{cases} 1, & \big||f|-f_c\big| < B_X/2, \\ 0, & \text{elsewhere.} \end{cases}$

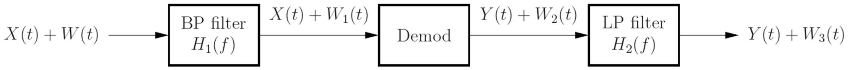
Filtered noise PSD: $R_{W_1}(f) = |H_1(f)|^2 R_W(f) = \begin{cases} R_0, & \big||f|-f_c\big| < B_X/2, \\ 0, & \text{elsewhere,} \end{cases}$

Filtered noise power: $P_{W_1} = \int_{-\infty}^{\infty} R_{W_1}(f)\,df = 2\int_{f_c-B_X/2}^{f_c+B_X/2} R_0\,df = 2B_X R_0.$

Signal power: $P_X = \frac{A^2}{2}$

Input SNR: $\frac{P_X}{P_{W_1}} = \frac{A^2/2}{2B_X R_0} = \frac{A^2}{4B_X R_0}$

AWGN Impact on Phase Modulation



Output signal: $Aa\,m(t)$

Output signal power: $P_Y = A^2a^2P_M$

Can be shown: $R_{W_2}(f) = \begin{cases} 2R_0, & |f| < B_X/2, \\ 0, & \text{elsewhere,} \end{cases}$

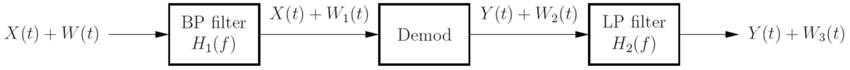
Filter: $H_2(f) = \begin{cases} 1, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$

Output noise: $R_{W_3}(f) = |H_2(f)|^2 R_{W_2}(f) = \begin{cases} 2R_0, & |f| < B, \\ 0, & \text{elsewhere,} \end{cases}$

Noise power: $P_{W_3} = \int_{-\infty}^{\infty} R_{W_3}(f)\,df = \int_{-B}^{B} 2R_0\,df = 4BR_0.$

Output SNR: $\frac{P_Y}{P_{W_3}} = \frac{A^2a^2P_M}{4BR_0}$

AWGN Impact on Frequency Modulation



Output signal: $Aa\,m(t)$

Output signal power: $P_Y = A^2a^2P_M$

Can be shown: $R_{W_2}(f) = \begin{cases} 2f^2R_0, & |f| < B_X/2, \\ 0, & \text{elsewhere,} \end{cases}$

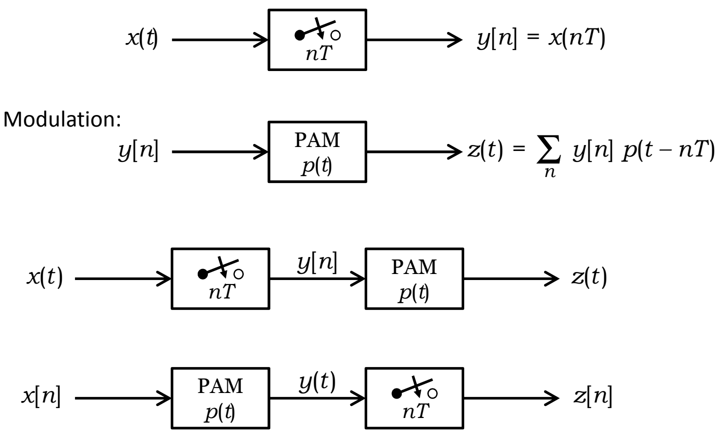
Filter: $H_2(f) = \begin{cases} 1, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$

Output noise: $R_{W_3}(f) = \begin{cases} 2f^2R_0, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$

Noise power: $P_{W_3} = \frac{4B^3R_0}{3}.$

Output SNR: $\frac{P_Y}{P_{W_3}} = \frac{3A^2a^2P_M}{4B^3R_0},$

Linear Mappings



Sampling: $x(t) \longrightarrow \boxed{nT} \longrightarrow y[n] = x(nT)$

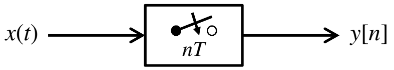
Pulse-Amplitude Modulation:
(PAM) $y[n] \longrightarrow \boxed{\text{PAM } p(t)} \longrightarrow z(t) = \sum_n y[n]\, p(t - nT)$

Reconstruction:

$t \to n \to t$: $x(t) \longrightarrow \boxed{nT} \xrightarrow{y[n]} \boxed{\text{PAM } p(t)} \longrightarrow z(t)$

$n \to t \to n$: $x[n] \longrightarrow \boxed{\text{PAM } p(t)} \xrightarrow{y(t)} \boxed{nT} \longrightarrow z[n]$

Sampling – Deterministically (from S&S)



$x(t) \longrightarrow \boxed{nT} \longrightarrow y[n]$

Sampling frequency: $f_s = 1/T$

Time domain: $y[n] = x(nT)$

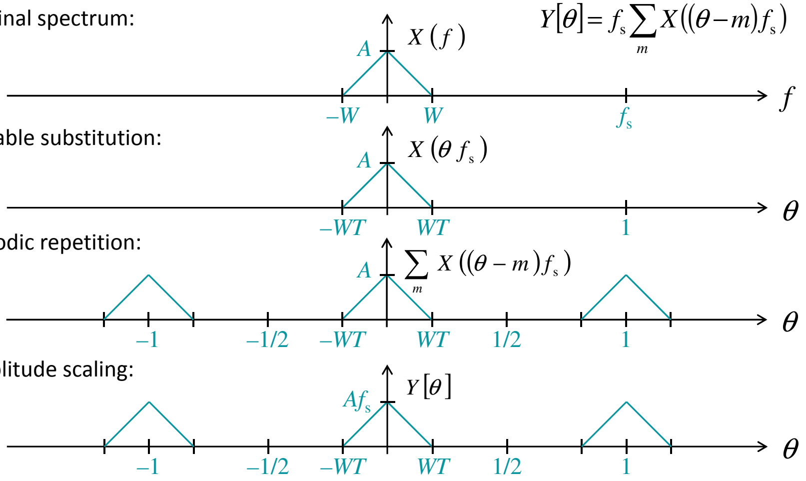
Frequency domain:
$$Y[\theta] = \frac{1}{T}\sum_m X\left(\frac{\theta - m}{T}\right) = f_s \sum_m X\big((\theta - m) f_s\big)$$

If $X(f) = 0$ **for** $|\, f \,| \geq f_s/2$:

$Y[\theta] = f_s X(\theta f_s)$ for $|\,\theta\,| < 1/2$

$Y[\theta] = Y[\theta + k]$ for k integer

Sampling – Frequency Domain



$$Y[\theta] = f_s \sum_m X\big((\theta - m) f_s\big)$$

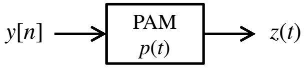
Original spectrum:

Variable substitution:

Periodic repetition:

Amplitude scaling:

PAM – Deterministically (from S&S)



$y[n] \longrightarrow \boxed{\text{PAM } p(t)} \longrightarrow z(t)$

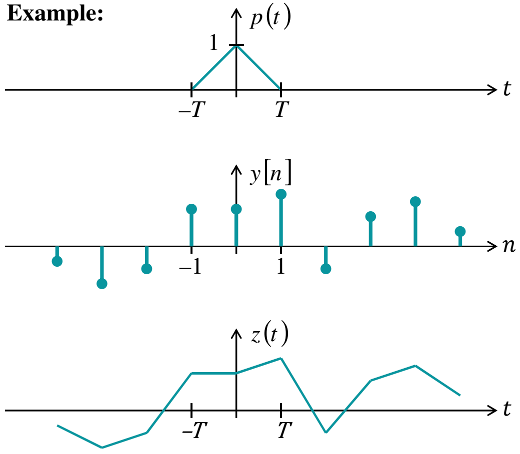
Time domain:

$$z(t) = \sum_n y[n]\, p(t - nT)$$

Frequency domain:

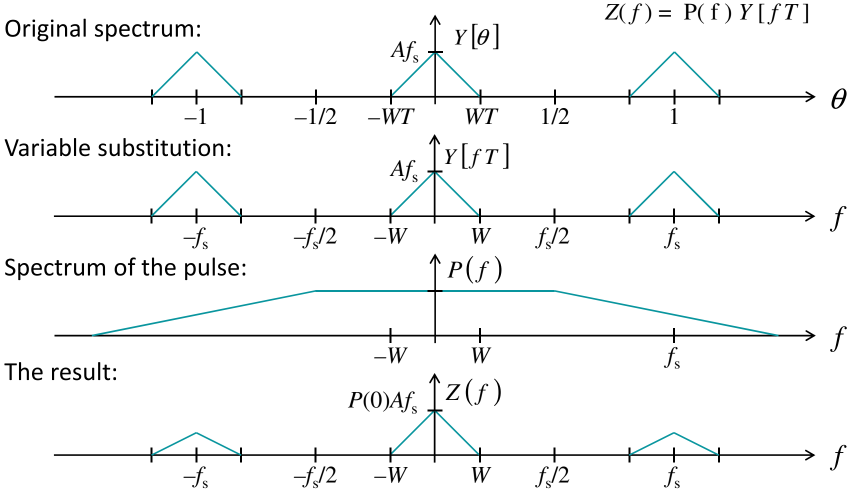
$$Z(f) = \text{P}(f)\, Y[fT]$$

Example:

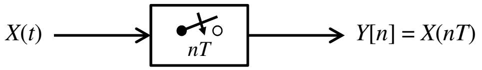


PAM – Frequency Domain

$$Z(f) = P(f)\, Y[fT]$$



Sampling of a Stochastic Process 1(2)



Assumption: $X(t)$ WSS

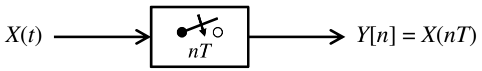
Mean: $m_Y[n] \overset{\text{Definition}}{=} E\{Y[n]\} \overset{\text{Sampling}}{=} E\{X(nT)\} \overset{X(t)\text{ WSS}}{=} m_X$

ACF: $r_Y[n, n+k] = E\{Y[n]Y[n+k]\} = E\{X(nT)X((n+k)T)\} = r_X(kT)$

This is sampling of the deterministic function $r_X(\tau)$.

PSD: $$R_Y[\theta] = \frac{1}{T}\sum_m R_X\left(\frac{\theta - m}{T}\right) = f_s \sum_m R_X((\theta - m)f_s)$$

Sampling of a Stochastic Process 2(2)



Assumption: $X(t)$ WSS

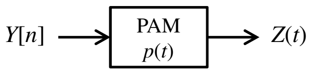
$$R_Y[\theta] = \frac{1}{T}\sum_m R_X\left(\frac{\theta - m}{T}\right) = f_s \sum_m R_X((\theta - m)f_s)$$

If $R_X(f) = 0$ for $|f| \geq f_s/2$:

$R_Y[\theta] = f_s R_X(\theta f_s)$ for $|\theta| < 1/2$

$R_Y[\theta] = R_Y[\theta + k]$ for k integer

PAM of a Stochastic Process 1(3)



Time domain (naïve approach):

$$Z(t) = \sum_n Y[n]\, p(t - nT)$$

Problem: Does not retain WSS.

$$m_Z(t) = m_Y \sum_n p(t - nT)$$

$$r_Z(t+\tau) = \\ = \sum_n \sum_m p(t-nT)\, p(t+\tau-mT)\, r_Y[n-m]$$

Time domain (alternative approach): Ψ unif. $[0,T]$

$Z(t) = \sum_n Y[n]\, p(t - nT - \Psi)$ Ψ & Y[n] indep.

Mean:

$$m_Z(t) = E\{Z(t)\} = E\left\{\sum_n Y[n] p(t - nT - \Psi)\right\}$$

$$= \sum_n E\{Y[n]\} E\{p(t - nT - \Psi)\} \quad \text{(indep.)}$$

$$= \sum_n m_Y \int_{-\infty}^{\infty} p(t - nT - \psi) f_\Psi(\psi)\, d\psi$$

$$= m_Y \sum_n \int_0^T p(t - nT - \psi) \frac{1}{T} d\psi = \Big/ \begin{matrix} \phi = t - nT - \psi \\ d\phi = -d\psi \end{matrix} \Big/$$

$$= m_Y \frac{1}{T} \sum_n \int_{t-nT-T}^{t-nT} p(\phi)\, d\phi = m_Y \frac{1}{T} \int_{-\infty}^{\infty} p(\phi)\, d\phi$$

$$= \frac{1}{T} P(0) m_Y$$

Independent of t.

PAM of a Stochastic Process 2(3)

ACF:

$$r_Z(t+\tau,t)=\mathrm{E}\{Z(t+\tau)Z(t)\}=\mathrm{E}\left\{\sum_n Y[n]p(t+\tau-nT-\Psi)\sum_m Y[m]p(t-mT-\Psi)\right\}$$

$$=\sum_n\sum_m \mathrm{E}\{Y[n]Y[m]\}\mathrm{E}\{p(t+\tau-nT-\Psi)p(t-mT-\Psi)\}$$

$$=\sum_n\sum_m r_Y[n-m]\int_0^T p(t+\tau-nT-\psi)p(t-mT-\psi)\frac{1}{T}d\psi = \Big/\ k=n-m\ \Big/$$

$$=\frac{1}{T}\sum_k r_Y[k]\sum_m\int_0^T p(t+\tau-(k+m)T-\psi)p(t-mT-\psi)d\psi = \Big/ \begin{matrix}\phi=\psi+mT-t\\ d\phi=d\psi\end{matrix} \Big/$$

$$=\frac{1}{T}\sum_k r_Y[k]\sum_m\int_{mT-t}^{T+mT-t} p(\tau-kT-\phi)p(-\phi)d\phi = \Big/\ \text{define}\ \ \tilde{p}(t)=p(-t)\ \Big/$$

$$=\frac{1}{T}\sum_k r_Y[k]\int_{-\infty}^{\infty} p(\tau-kT-\phi)\tilde{p}(\phi)d\phi=\frac{1}{T}\sum_k r_Y[k](p*\tilde{p})(\tau-kT)$$

Independent of *t*.

PAM of a Stochastic Process 3(3)

Mean: $m_Z=\frac{1}{T}P(0)m_Y$

ACF: $r_Z(\tau)=\frac{1}{T}\sum_k r_Y[k](p*\tilde{p})(\tau-kT)$

PSD: $R_Z(f)=\mathcal{F}\left\{\frac{1}{T}\sum_k r_Y[k](p*\tilde{p})(\tau-kT)\right\}$

Independent of *t*.
Thus WSS

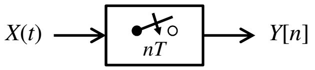
Notice: The sum is PAM of $r_Y[k]$ using pulse $(p*\tilde{p})(\tau)$.

$$\mathcal{F}\{(p*\tilde{p})(\tau)\}=|P(f)|^2$$

Recall deterministic PAM:

$$R_Z(f)=\frac{1}{T}|P(f)|^2R_Y[fT]$$

Sampling and PAM of WSS Processes – Summary

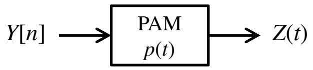


$Y[n]=X(nT)$

$m_Y=m_X$

$r_Y[k]=r_X(kT)$

$R_Y[\theta]=\frac{1}{T}\sum_m R_X\left(\frac{\theta-m}{T}\right)$



$Z(t)=\sum_n Y[n]\,p(t-kT-\Psi)$

$m_Z=\frac{1}{T}P(0)m_Y$

Ψ unif. $[0,T]$
Ψ & Y[n] indep.

$r_Z(\tau)=\frac{1}{T}\sum_k r_Y[k](p*\tilde{p})(\tau-kT)$

$R_Z(f)=\frac{1}{T}|P(f)|^2R_Y[fT]$

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