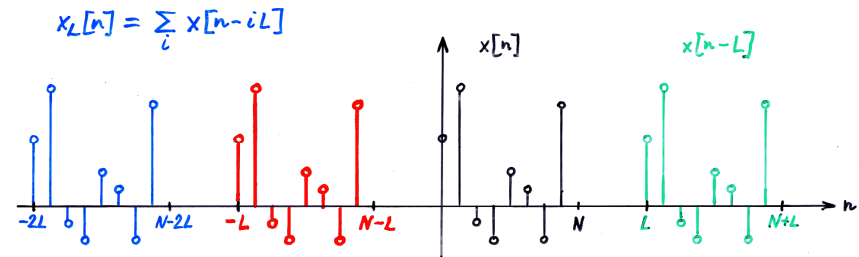


TSDT14 Signal Theory

Lecture 4 Estimation

Mikael Olofsson
Department of EE (ISY)
Div. of Communication Systems

DFT – Avoiding Aliasing



If $L < N$, then we get overlap and aliasing in the time domain.

Therefore: Demand $L \geq N$.

Note:
$$X_L[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{L} n} = \sum_{n=0}^{L-1} x_L[n] e^{-j2\pi \frac{k}{L} n}$$

DFT – Signal Analysis

Time-discrete signal with limited duration:

$$x[n] = 0 \text{ for } n \notin \{0, 1, \dots, N-1\}$$

Fourier transform:
$$X[\theta] = \sum_n x[n] e^{-j2\pi \theta n} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \theta n} \text{ cont. w. period 1.}$$

DFT of length L :
$$X_L[k] = X[k/L] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{L} n} \text{ for } k \in \{0, 1, \dots, L-1\}$$

IDFT (inverse):
$$x_L[n] = \frac{1}{L} \sum_{k=0}^{L-1} X_L[k] e^{j2\pi \frac{k}{L} n} \Rightarrow x_L[n+L] = x_L[n] \text{ since } e^{j2\pi \frac{k}{L} L} = e^{j2\pi k} = 1$$

Relation to $x[n]$:
$$x_L[n] = \frac{1}{L} \sum_{k=0}^{L-1} \sum_{m=0}^{N-1} x[m] e^{-j2\pi \frac{k}{L} m} e^{j2\pi \frac{k}{L} n} = \sum_{m=0}^{N-1} x[m] \cdot \frac{1}{L} \sum_{k=0}^{L-1} e^{-j2\pi \frac{k}{L} (m-n)} = \sum_{i=-\infty}^{\infty} x[n-iL] = \begin{cases} x[n], & m-n = 0 \text{ mod } L \\ 0, & \text{elsewhere} \end{cases}$$

DFT – Periodic Convolution

We are used to: $y[n] = (x * h)[n] \Leftrightarrow Y[\theta] = X[\theta] \cdot H[\theta]$

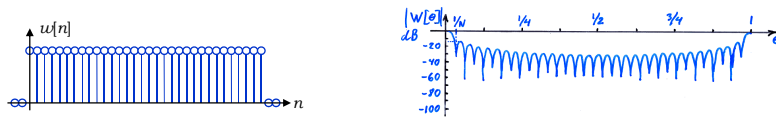
But we have: $y[n] = x[n] \cdot h[n] \Leftrightarrow Y[\theta] = \int_0^1 X[\theta] H[\theta - \theta] d\theta$

With DFT:
$$\begin{aligned} X_L[k] &= X_L[k] \cdot H_L[k] \Leftrightarrow \\ y_L[n] &= \text{IDFT}\{X_L[k] H_L[k]\} = \frac{1}{L} \sum_{k=0}^{L-1} X_L[k] H_L[k] e^{j2\pi \frac{k}{L} n} \\ &= \frac{1}{L} \sum_{k=0}^{L-1} X_L[k] \cdot \sum_{m=0}^{L-1} h_L[m] e^{-j2\pi \frac{k}{L} m} e^{j2\pi \frac{k}{L} n} \\ &= \sum_{m=0}^{L-1} h_L[m] \cdot \underbrace{\frac{1}{L} \sum_{k=0}^{L-1} X_L[k] e^{j2\pi \frac{k}{L} (n-m)}}_{x_L[n-m]} \\ &= \sum_{m=0}^{L-1} h_L[m] x_L[n-m] \end{aligned}$$

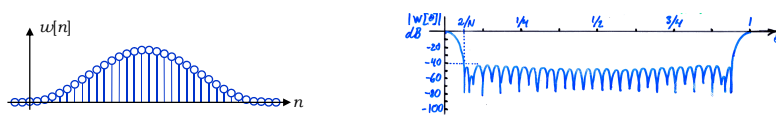
And also:
$$y_L[n] = x_L[n] \cdot h_L[n] \Leftrightarrow Y_L[k] = \frac{1}{L} \sum_{m=0}^{L-1} X_L[m] H_L[k-m]$$

Examples of Windows, $N=32$

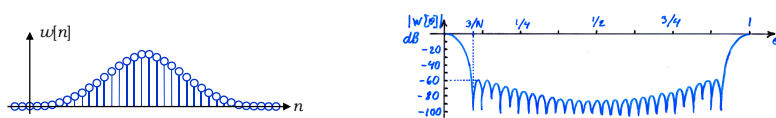
Rectangular window: $w[n] = 1, n \in \{0, 1, \dots, N-1\}$



Hamming window: $w[n] = 0.54 - 0.46 \cos(\frac{2\pi n}{N-1}), n \in \{0, 1, \dots, N-1\}$



Blackman window: $w[n] = 0.42 - 0.5 \cos(\frac{2\pi n}{N-1}) + 0.08 \cos(\frac{4\pi n}{N-1}), n \in \{0, 1, \dots, N-1\}$



Ergodicity again

A WSS process:

$$m_X = E\{X(t)\}$$

Time average of one realization:

$$m_T = \frac{1}{2T} \int_{-T}^T x(t) dt$$

Time average of the process:

$$M_T = \frac{1}{2T} \int_{-T}^T X(t) dt$$

$$E\{M_T\} = E\left\{\frac{1}{2T} \int_{-T}^T X(t) dt\right\} = \frac{1}{2T} \int_{-T}^T E\{X(t)\} dt = \frac{1}{2T} \int_{-T}^T m_X dt = m_X$$

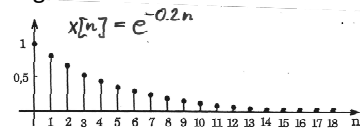
Definition: If $\lim_{T \rightarrow \infty} E\{(M_T - m_X)^2\} = 0$ then $X(t)$ is said to be ergodic with respect to the mean, and we write

$$m_X = \text{l.i.m.}_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) dt \quad (\text{limes in mean})$$

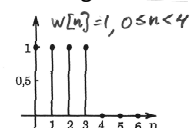
Interpretation: The time average of a realization is very close to the ensemble mean with a probability that is very close to 1 ($\rightarrow 1, T \rightarrow \infty$).

Using Windows

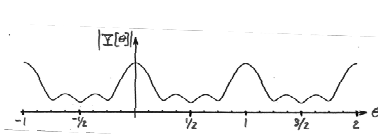
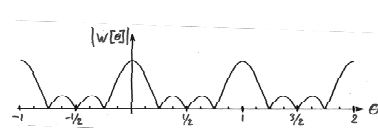
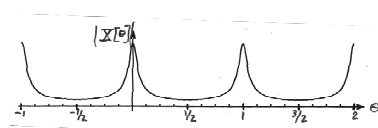
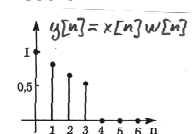
Signal:



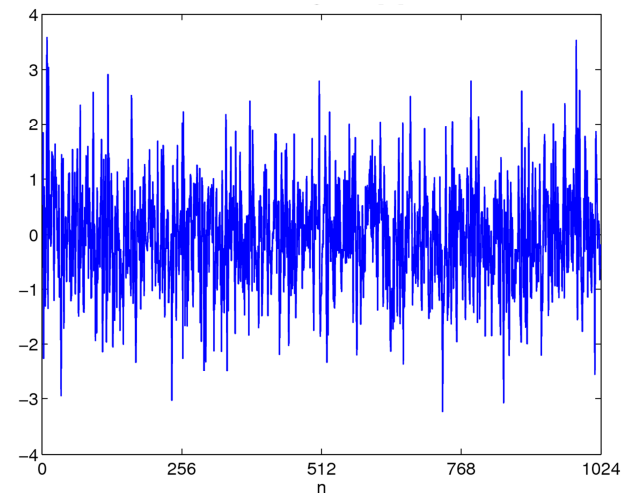
Rectangular window:



Result:



A Signal



Estimating a General Parameter

$X[n]$: An ergodic time-discrete process, with realization $x[n]$.

Estimation of a general parameter a_X , i.e. some ensemble average:

$$\hat{a}_X = g(x[0], x[1], \dots, x[N-1])$$

Corresponding stochastic variable: $\hat{A}_X = g(X[0], X[1], \dots, X[N-1])$

Bias (difference from actual value): $B = E\{\hat{A}_X\} - a_X$

Unbiased estimate: $B = 0$

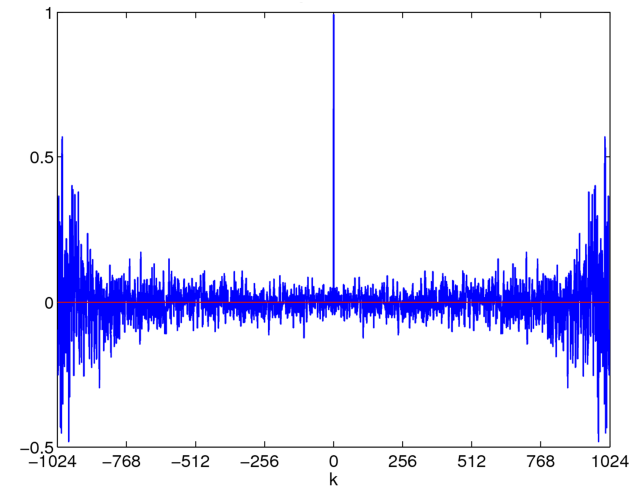
Asymptotically unbiased estimate: $B \rightarrow 0$ when $N \rightarrow \infty$

Variance: $\sigma_{\hat{A}_X}^2 = E\{\hat{A}_X^2\} - E^2\{\hat{A}_X\}$

Quadratic error: $\varepsilon^2 = \sigma_{\hat{A}_X}^2 + B^2$

Consistent estimate: $\varepsilon^2 \rightarrow 0$ when $N \rightarrow \infty$

Blackman-Tukey's Estimate of the ACF 2(2)



Blackman-Tukey's Estimate of the ACF 1(2)

ACF of a WSS process: $r_X[k] = E\{X[n+k]X[n]\}$

Blackman-Tukey's method: $\hat{r}_X[k] = \frac{1}{N-|k|} \sum_{n=0}^{N-|k|-1} x[n+|k|]x[n]$

Mean: $E\{\hat{r}_X[k]\} = r_X[k]$

Variance, assuming a Gaussian process with mean zero:

$$|k| < N: \sigma_{\hat{r}_X}^2[k] = \frac{1}{N-|k|} \sum_{|m| < N-|k|} \left(1 - \frac{1}{N-|k|}\right) (r_X^2[m] + r_X[m+k]r_X[m-k])$$

Fixed k : $\sigma_{\hat{r}_X}^2[k] \rightarrow 0$ when $N \rightarrow \infty$

Problem: Large k , Ex. $k = N-1$ $\sigma_{\hat{r}_X}^2[k] \rightarrow r_X^2[0]$ when $N \rightarrow \infty$

Bartlett's Estimate of the ACF 1(2)

Recall Blackman-Tukey: $\hat{r}_X[k] = \frac{1}{N-|k|} \sum_{n=0}^{N-|k|-1} x[n+|k|]x[n]$

Bartlett's method: $\hat{r}_X[k] = \frac{1}{N} \sum_{n=0}^{N-|k|-1} x[n+|k|]x[n]$

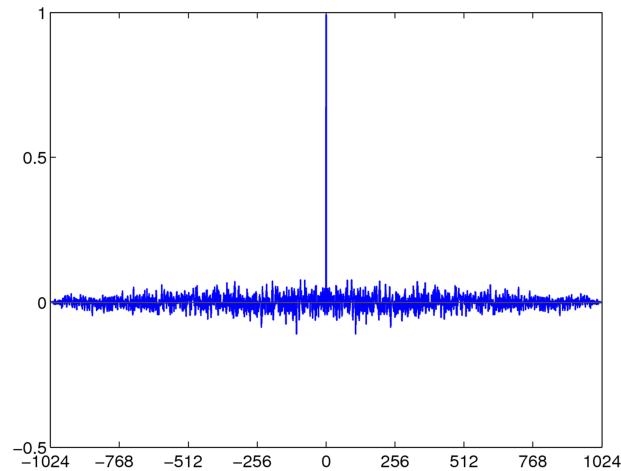
Mean: $E\{\hat{r}_X[k]\} = \left(1 - \frac{|k|}{N}\right) r_X[k] \rightarrow r_X[k]$ when $N \rightarrow \infty$

Variance, assuming a Gaussian process with mean zero:

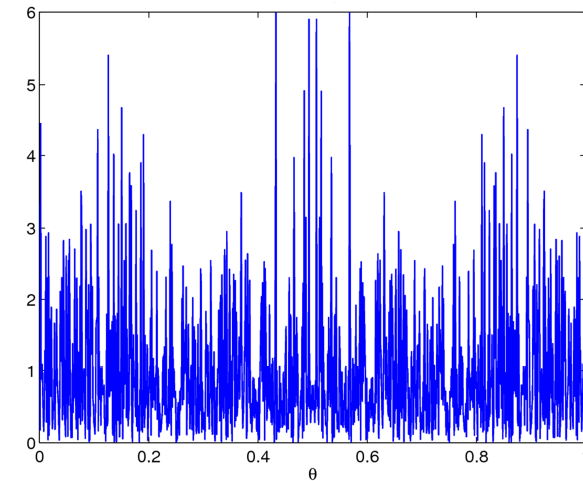
$$|k| < N: \sigma_{\hat{r}_X}^2[k] = \frac{1}{N} \sum_{|m| < N-|k|} \left(1 - \frac{|k|+|m|}{N}\right) (r_X^2[m] + r_X[m+k]r_X[m-k])$$

All k : $\sigma_{\hat{r}_X}^2[k] \rightarrow 0$ when $N \rightarrow \infty$

Bartlett's Estimate of the ACF 2(2)



Periodogram



Estimating the PSD 1(2)

PSD of a WSS process:

$$R_X[\theta] = \mathcal{F}\{r_X[k]\}$$

Observed sequence:

$$x[n]$$

Considered part of sequence:

$$x_N[n] = \begin{cases} x[n], & 0 \leq n < N \\ 0 & \text{elsewhere} \end{cases}$$

Bartlett's estimate of the ACF: $\hat{r}_X[k] = \frac{1}{N} \sum_{m=-\infty}^{\infty} x_N[m+k]x_N[m]$

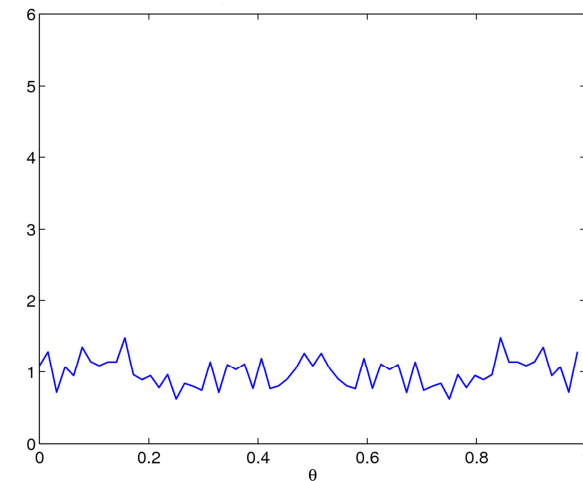
Estimate of the PSD:

$$\hat{R}_X[\theta] = \mathcal{F}\{\hat{r}_X[k]\}$$

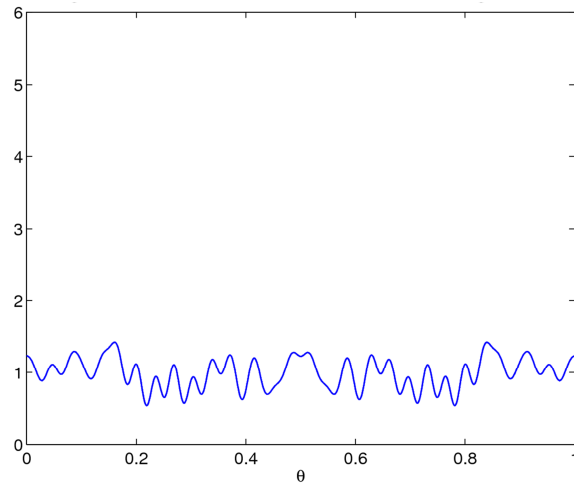
$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} \frac{1}{N} \sum_{m=-\infty}^{\infty} x_N[m+k]x_N[m]e^{-j2\pi\theta k} = \frac{1}{N} \sum_{m=-\infty}^{\infty} x_N[m] \sum_{k=-\infty}^{\infty} x_N[m+k]e^{-j2\pi\theta k} \\ &= \frac{1}{N} X_N[\theta] \sum_{m=-\infty}^{\infty} x_N[m]e^{j2\pi\theta m} = \frac{1}{N} X_N[\theta]X_N^*[\theta] = \frac{1}{N} |X_N[\theta]|^2 \end{aligned}$$

Called periodogram

Averaged Periodogram

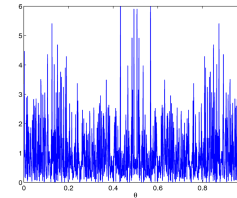


Estimated PSD using Rectangular Window

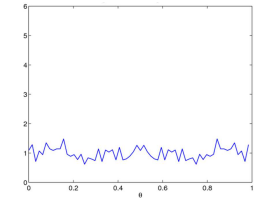


Smoothing – Overview

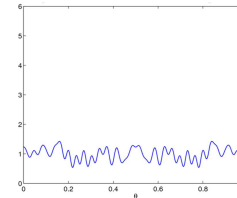
Raw periodogram



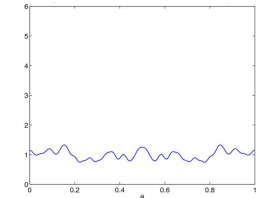
Averaged periodograms



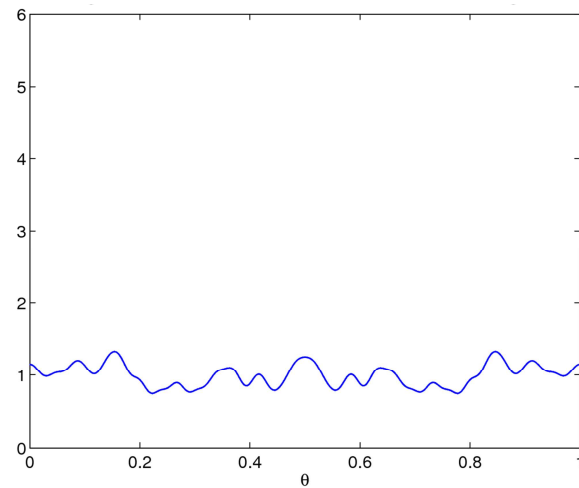
Smoothing - Rectangular



Smoothing - Hamming



Estimated PSD using Hamming Window



Mikael Olofsson
ISY/CommSys

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