TSDT14 Signal Theory

Lecture 1
Introduction and Repetition

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TSDT14 Signal Theory - Formalia

Information & course material: lisam.liu.se

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Examination:

Laborations (2 hp): Study 1, 2, 3, 4 (4×2 hours)
Sign-up on the web
Report before end of exam period

Written exam (4 hp): 1 simple task – Demand: 2/3 OK
5 tasks (5 points each), max 25
Pass: 10 points

Course Aims 1(2)

After passing the course, the student should

- be able to clearly define central concepts regarding stochastic processes, using own words. ***(task 1)***
- be able to reliably perform standard calculations regarding stochastic processes, e.g. LTI filtering (both time continuous and time discrete), sampling and pulse amplitude modulation. ***(task 1)***
- be able to reliably perform standard calculations regarding stochastic processes being exposed to certain momentary non-linearities that are common in telecommunication, especially uniform quantization and monomial non-linearities of low degrees. ***(task 1)***
- with some reliability be able to solve problems that demand integration of knowledge from different parts of the course. ***(tasks 2-6)***

Course Aims 2(2)

After passing the course, the student should

- be able to account for the connection between different concepts in the course in a structured way using adequate terminology. ***(lab report)***
- be able to estimate the auto correlation function and power spectral density of a stochastic process based on a realization of the process. Also, clearly and logically account for those estimations and conclusions that can be drawn from them. ***(lab report)***

Tutorial Sessions (Teaching Sessions)

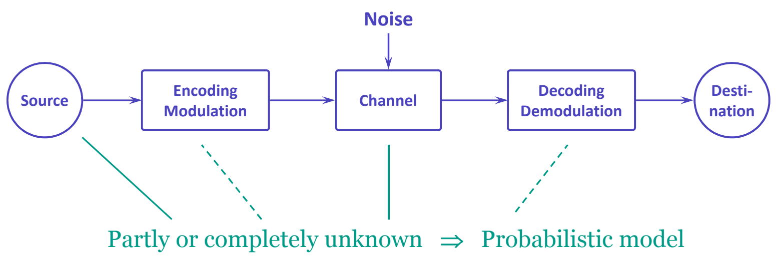
All tutorial are given in English.

- Group A: Unnikrishnan Kunnath Ganesan
- Group B: Amin Ghazanfari
- Group C: Özgecan Özdogan

You are free to follow any tutorial group you wish. Please try to spread out over the groups.

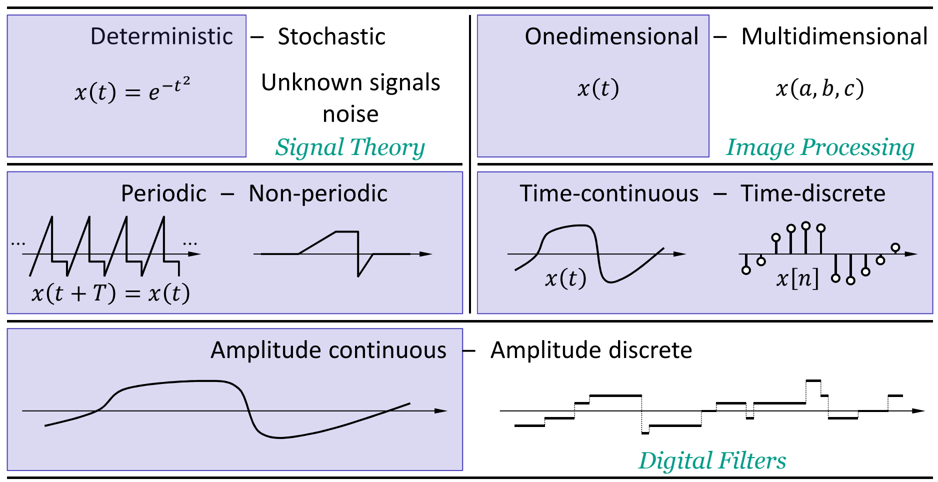
Individual teaching can be done in any language that works.

Standard Situation



We can have
Linear and non-linear filtering
Sampling and reconstruction
Up- and down-sampling
Modulation
Also: Error Correction, Packing, Cryptology,...

Classification of Signals



Classification of Systems

Linearity – Preserves linear combinations.

Time-invariance – Behaves the same way all the time.

Causality – Has no knowledge about the future.

Stability – Bounded input results in bounded output.

LTI – Linear and Time-Invariant. Convolution.

Frequency Domain

Jean Baptiste Joseph Fourier
1768 – 1830

Time-Continuous Fourier transform:

Signal: $x(t)$

Spectrum: $X(f) = F\{x(t)\} = \int_{-\infty}^{\infty} x\,(t) e^{-j2\pi f t} dt$

Inverse: $x(t) = F^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X\,(f) e^{j2\pi f t} df$

Time-Discrete Fourier transform:

Signal: $x[n]$

Spectrum: $X[\theta] = F\{x[n]\} = \sum_n x[n] e^{-j2\pi\theta n}$

Inverse: $x[n] = F^{-1}\{X[\theta]\} = \int_0^1 X\,[\theta] e^{j2\pi\theta n} d\theta$

Signal Power and Signal Energy – Parseval

Signal power: $|x(t)|^2$ Signal energy: $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$

Parseval's relation (special case):

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) x^*(t) \, dt = \int_{-\infty}^{\infty} X(f) X^*(f) \, df = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Energy spectrum: $|X(f)|^2$

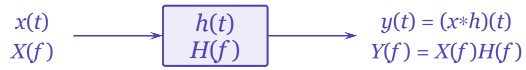
Parseval's relation (generally): $\int_{-\infty}^{\infty} a(t) b^*(t) \, dt = \int_{-\infty}^{\infty} A(f) B^*(f) \, df$

Output from an LTI System

Notation: $A(f) = \mathscr{F}\{a(t)\}$ $B(f) = \mathscr{F}\{b(t)\}$

Property: $\mathscr{F}\{(a*b)(t)\} = \int_{-\infty}^{\infty} (a*b)(t) \, e^{-j2\pi f t} \, dt = A(f)B(f)$

LTI System:



The Principle sine in – sine out

For LTI systems, we have:

Sine in – Sine out (the same frequency)

More precisely:

Input: $x(t) = \hat{X} \sin(2\pi f_0 t + \varphi)$

Output: $y(t) = \hat{X} |H(f_0)| \sin(2\pi f_0 t + \varphi + \arg\{H(f_0)\})$

Amplitude characteristic: $|H(f)|$

Phase characteristic: $\arg\{H(f)\}$

This is the $j\omega$ method in condensed form.

Classification of Frequency Selective Filters

- A (frequency selective) filter is an LTI system. Usually it lets some frequency band through or stops it.
- Usually an electrical network – either passive or active.

Notation | Ideal amplitude characteristics | Real amplitude char.

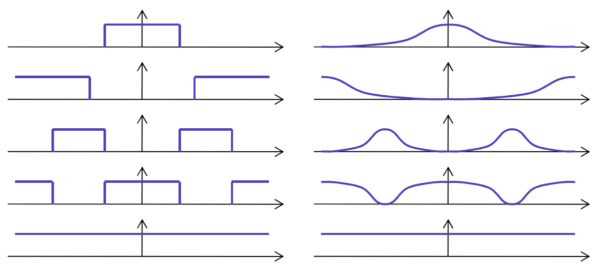
Lowpass filter (LP filter):

Highpass filter (HP filter):

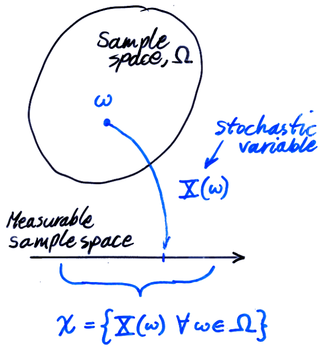
Bandpass filter (BP filter):

Bandstop filter (BS filter):

Allpass filter (AP filter):



Probabilities and Distributions



Probability: $\Pr\{A\} \in [0,1]$

Joint prob.: $\Pr\{A,B\}$

Cond. Prob.: $\Pr\{A \mid B\} = \dfrac{\Pr\{A,B\}}{\Pr\{B\}}$

Prob. distr.: $F_X(x) = \Pr\{X \le x\} \in [0,1]$

Prob. density.: $f_X(x) = \dfrac{d}{dx} F_X(x)$

Properties: $F_X(x)$ is non-decreasing

$f_X(x) \ge 0$ for all x

$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$\Pr\{x_1 < X \le x_2\} = \int_{x_1+}^{x_2+} f_X(x)\, dx$

Example

Game based on tossing two fair coins:

2 heads	+400
2 tails	–100
1 tail, 1 head	–200

$\Omega = \{HH, HT, TH, TT\}$

$\Omega_X = \{-200, -100, 400\}$

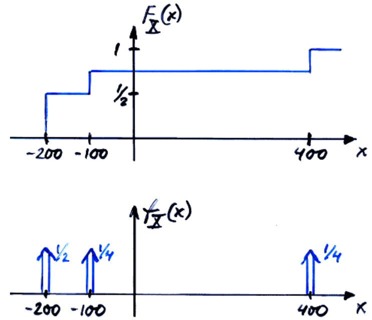
$$X(\omega) = \begin{cases} 400, & \omega = HH \\ -100, & \omega = TT \\ -200, & \omega \in \{HT, TH\} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < -200 \\ 1/2, & -200 \le x < -100 \\ 3/4, & -100 \le x < 400 \\ 1, & x \ge 400 \end{cases}$$

$= \frac{1}{2}u(x+200) + \frac{1}{4}u(x+100) + \frac{1}{4}u(x-400)$

$f_X(x) = \frac{d}{dx} F_X(x)$

$= \frac{1}{2}\delta(x+200) + \frac{1}{4}\delta(x+100) + \frac{1}{4}\delta(x-400)$



Expectations

Expectation (mean):

$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\, dx$

For discrete variables also:

$E\{X\} = \sum_i x_i \cdot \Pr\{X = x_i\}$

For functions of a variable:

$Y = g(X)$

$E\{Y\} = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$

$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

Also:

m_X, m_Y

Quadratic mean (Power):

$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$

Variance:

$Var\{X\} = E\{(X - m_X)^2\}$

$= E\{X^2\} - m_X^2$

Also:

σ_X^2

Standard deviation:

σ_X

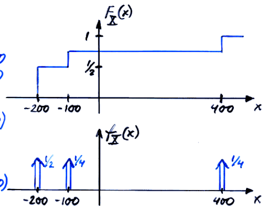
Example cont'd

$$F_X(x) = \begin{cases} 0, & x < -200 \\ 1/2, & -200 \le x < -100 \\ 3/4, & -100 \le x < 400 \\ 1, & x \ge 400 \end{cases}$$

$$= \tfrac{1}{2}u(x+200) + \tfrac{1}{4}u(x+100) + \tfrac{1}{4}u(x-400)$$

$$f_X(x) = \frac{d}{dx}F_X(x)$$

$$= \tfrac{1}{2}\delta(x+200) + \tfrac{1}{4}\delta(x+100) + \tfrac{1}{4}\delta(x-400)$$



$Y = \operatorname{sgn} X$

$\Pr\{Y=1\} = \Pr\{X>0\} = \Pr\{X=400\} = 1/4$

$\Pr\{Y=-1\} = \Pr\{X<0\} = \Pr\{X=-100\} + \Pr\{X=-200\} = 3/4$

$F_Y(y) = \tfrac{3}{4}u(y+1) + \tfrac{1}{4}u(y-1)$

$f_Y(y) = \tfrac{3}{4}\delta(y+1) + \tfrac{1}{4}\delta(y-1)$

$$E\{X\} = \int_{-\infty}^{\infty} x\left(\tfrac{1}{2}\delta(x+200) + \tfrac{1}{4}\delta(x+100) + \tfrac{1}{4}\delta(x-400)\right)dx$$

$$= -200\cdot\tfrac{1}{2} - 100\cdot\tfrac{1}{4} + 400\cdot\tfrac{1}{4} = -25$$

$$E\{X^2\} = (-200)^2\cdot\tfrac{1}{2} + (-100)^2\cdot\tfrac{1}{4} + (400)^2\cdot\tfrac{1}{4} = 62500$$

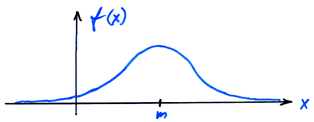
$$Var\{X\} = E\{X^2\} - (E\{X\})^2 = 61875$$

Gaussian Distributions, $N(m,\sigma)$

Probability density:

$$f(x) = \frac{1}{\sigma\cdot\sqrt{2\pi}}\cdot e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Probability distribution is hard.



Mean: m

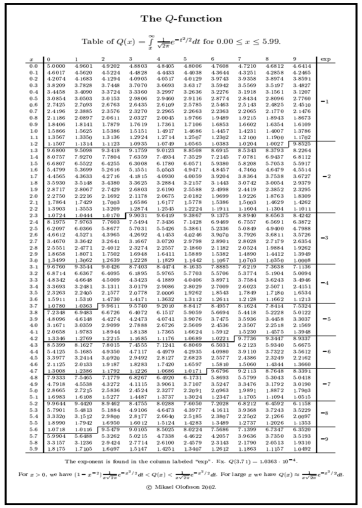
Variance: σ^2

The Q function:

$Q(x) = 1 - F(x)$ for $N(0,1)$

$= \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$

From a table



Example of the Q Function

x	0	1	2	3	4	5	6	7	8	9	exp
0.0	5.0000	4.9601	4.9202	4.8803	4.8405	4.8006	4.7608	4.7210	4.6812	4.6414	
0.1	4.6017	4.5620	4.5224	4.4828	4.4433	4.4038	4.3644	4.3251	4.2858	4.2465	
0.2	4.2074	4.1683	4.1294	4.0905	4.0517	4.0129	3.9743	3.9358	3.8974	3.8591	
0.3	3.8209	3.7828	3.7448	3.7070	3.6693	3.6317	3.5942	3.5569	3.5197	3.4827	
0.4	3.4458	3.4090	3.3724	3.3360	3.2997	3.2636	3.2276	3.1918	3.1561	3.1207	
0.5	3.0854	3.0503	3.0153	2.9806	2.9460	2.9116	2.8774	2.8434	2.8096	2.7760	
0.6	2.7425	2.7093	2.6763	2.6435	2.6109	2.5785	2.5463	2.5143	2.4825	2.4510	−1
0.7	2.4196	2.3885	2.3576	2.3270	2.2965	2.2663	2.2363	2.2065	2.1770	2.1476	
0.8	2.1186	2.0897	2.0611	2.0327	2.0045	1.9766	1.9489	1.9215	1.8943	1.8673	
0.9	1.8406	1.8141	1.7879	1.7619	1.7361	1.7106	1.6853	1.6602	1.6354	1.6109	
1.0	1.5866	1.5625	1.5386	1.5151	1.4917	1.4686	1.4457	1.4231	1.4007	1.3786	
1.1	1.3567	1.3350	1.3136	1.2924	1.2714	1.2507	1.2302	1.2100	1.1900	1.1702	
1.2	1.1507	1.1314	1.1123	1.0935	1.0749	1.0565	1.0383	1.0204	1.0027	9.8525	
1.3	9.6800	9.5098	9.3418	9.1759	9.0123	8.8508	8.6915	8.5343	8.3793	8.2264	
1.4	8.0757	7.9270	7.7804	7.6359	7.4934	7.3529	7.2145	7.0781	6.9437	6.8112	
1.5	6.6807	6.5522	6.4255	6.3008	6.1780	6.0571	5.9380	5.8208	5.7053	5.5917	
1.6	5.4799	5.3699	5.2616	5.1551	5.0503	4.9471	4.8457	4.7460	4.6479	4.5514	
1.7	4.4565	4.3633	4.2716	4.1815	4.0930	4.0059	3.9204	3.8364	3.7538	3.6727	−2
1.8	3.5930	3.5148	3.4380	3.3625	3.2884	3.2157	3.1443	3.0742	3.0054	2.9379	
1.9	2.8717	2.8067	2.7429	2.6803	2.6190	2.5588	2.4998	2.4419	2.3852	2.3295	
2.0	2.2750	2.2216	2.1692	2.1178	2.0675	2.0182	1.9699	1.9226	1.8763	1.8309	
2.1	1.7864	1.7429	1.7003	1.6586	1.6177	1.5778	1.5386	1.5003	1.4629	1.4262	
2.2	1.3903	1.3553	1.3209	1.2874	1.2545	1.2224	1.1911	1.1604	1.1304	1.1011	
2.3	1.0724	1.0444	1.0170	9.9031	9.6419	9.3867	9.1375	8.8940	8.6563	8.4242	

$Q(\underline{1.96}) \approx 2.4998\cdot 10^{-2}$

Example of the Q Function cont'd

x	0	1	2	3	4	5	6	7	8	9	exp
4.2	1.3346	1.2769	1.2215	1.1685	1.1176	1.0689	1.0221	9.7736	9.3447	8.9337	
4.3	8.5399	8.1627	7.8015	7.4555	7.1241	6.8069	6.5031	6.2123	5.9340	5.6675	
4.4	5.4125	5.1685	4.9350	4.7117	4.4979	4.2935	4.0980	3.9110	3.7322	3.5612	−6
4.5	3.3977	3.2414	3.0920	2.9492	2.8127	2.6823	2.5577	2.4386	2.3249	2.2162	
4.6	2.1125	2.0133	1.9187	1.8283	1.7420	1.6597	1.5810	1.5060	1.4344	1.3660	
4.7	1.3008	1.2386	1.1792	1.1226	1.0686	1.0171	9.6796	9.2113	8.7648	8.3391	
4.8	7.9333	7.5465	7.1779	6.8267	6.4920	6.1731	5.8693	5.5799	5.3043	5.0418	
4.9	4.7918	4.5538	4.3272	4.1115	3.9061	3.7107	3.5247	3.3476	3.1792	3.0190	−7
5.0	2.8665	2.7215	2.5836	2.4524	2.3277	2.2091	2.0963	1.9891	1.8872	1.7903	
5.1	1.6983	1.6108	1.5277	1.4487	1.3737	1.3024	1.2347	1.1705	1.1094	1.0515	

$Q(x) = 10^{-6} \quad \Rightarrow \quad x \approx 4.75$

For $x > 0$, we have $(1 - x^{-2})\frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt < Q(x) < \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt.$

For large x we have $Q(x) \approx \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt.$

Other Common Distributions

Uniform distribution:

$$f(x)=\begin{cases}\frac{1}{b-a}, & a\le x<b\\ 0, & \text{elsewhere}\end{cases}$$

Mean: $\frac{a+b}{2}$

Variance: $\frac{(b-a)^2}{12}$

Exponential distribution:

$$f(x)=\frac{1}{c}e^{-x/c}\cdot u(x)$$

Mean: c

Variance: c^2

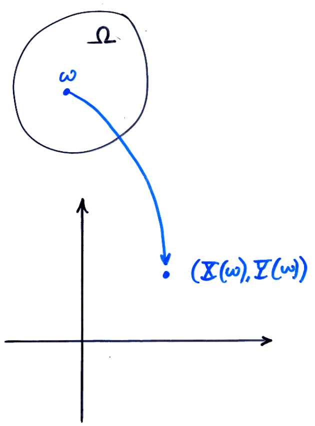
Binary distribution:

$$f(x)=p\cdot\delta(x-a)+(1-p)\cdot\delta(x-b)$$

Mean: $p\cdot a+(1-p)\cdot b$

Variance: $p(1-p)\cdot(b-a)^2$

Two-Dimensional Stochastic Variables



Distribution: $F_{X,Y}(x,y)=Pr\{X\le x, Y\le y\}$

Density: $f_{X,Y}(x,y)=\frac{\partial^2}{\partial x\,\partial y}F_{X,Y}(x,y)$

Properties: $f_{X,Y}(x,y)\ge 0 \quad \forall\, x,y$

$$\iint_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy=1$$

Marginal: $f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Function: $E\{g(X,Y)\}=$

$$=\iint_{-\infty}^{\infty} g(x,y)\,f_{X,Y}(x,y)\,dx\,dy$$

Dependencies

Definition: X & Y are independent if $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ holds.

Theorem: Independent $\Leftrightarrow$ $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ holds.

Definition: Covariance: $\mathrm{Cov}\{X,Y\} = \mathrm{E}\{(X-m_X)(Y-m_Y)\}$

Theorem: $\mathrm{Cov}\{X,Y\} = \mathrm{E}\{XY\} - m_X m_Y$

Definition: X & Y are uncorrelated if $\mathrm{Cov}\{X,Y\} = 0$ holds.

Theorem: Independent $\Rightarrow$ uncorrelated.

Note: $\mathrm{Var}\{X\} = \mathrm{Cov}\{X,X\}$

Theorem: Uncorrelated $\Leftrightarrow$ $\mathrm{E}\{XY\} = \mathrm{E}\{X\}\mathrm{E}\{Y\}$

$\Leftrightarrow$ $\mathrm{Var}\{X+Y\} = \mathrm{Var}\{X\} + \mathrm{Var}\{Y\}$

Bayes' Rule

Conditional distribution: $F_{X|Y}(x|y)=\frac{\int_{-\infty}^{x} f_{X,Y}(z,y)\,dz}{f_Y(y)}$

Conditional density: $f_{X|Y}(x|y)=\frac{d}{dx}F_{X,Y}(x,y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}$

Bayes rule (continuous): $f_{X|Y}(x|y)=\frac{f_{Y|X}(y|x)}{f_Y(y)}\cdot f_X(x)$

(discrete): $Pr\{X=x|Y=y\}=\frac{Pr\{Y=y|X=x\}}{Pr\{Y=y\}}\cdot Pr\{X=x\}$

(Mixed): $Pr\{X=x|Y=y\}=\frac{f_{Y|X}(y|x)}{f_Y(y)}\cdot Pr\{X=x\}$

Multi-Dimensional Stochastic Variables

Distribution: $F_{\mathbf{X}_1,\dots,\mathbf{X}_N}(x_1,\dots,x_N) = Pr\{\mathbf{X}_1 \le x_1, \dots, \mathbf{X}_N \le x_N\}$

Density: $f_{\mathbf{X}_1,\dots,\mathbf{X}_N}(x_1,\dots,x_N) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N} F_{\mathbf{X}_1,\dots,\mathbf{X}_N}(x_1,\dots,x_N)$

Vector notation: $\bar{\mathbf{X}} = (\mathbf{X}_1,\dots,\mathbf{X}_N)$, $\bar{x} = (x_1,\dots,x_N)$, $F_{\bar{\mathbf{X}}}(\bar{x})$, $f_{\bar{\mathbf{X}}}(\bar{x})$

You might think that this makes sense:

Independence: $F_{\bar{\mathbf{X}}}(\bar{x}) = \prod_{i=1}^{N} F_{\mathbf{X}_i}(x_i)$ & $f_{\bar{\mathbf{X}}}(\bar{x}) = \prod_{i=1}^{N} f_{\mathbf{X}_i}(x_i)$

Uncorrelated: $E\{\prod_{i=1}^{N} \mathbf{X}_i\} = \prod_{i=1}^{N} E\{\mathbf{X}_i\}$

$Var\{\sum_{i=1}^{N} \mathbf{X}_i\} = \sum_{i=1}^{N} Var\{\mathbf{X}_i\}$

Not entirely true. See course book for details (p. 58-59)

Jointly Gaussian Variables

$\bar{\mathbf{X}} = (\mathbf{X}_1,\dots,\mathbf{X}_N)$ *is called Jointly Gaussian if the following holds:*

$$f_{\bar{\mathbf{X}}}(\bar{x}) = \frac{1}{(2\pi)^{N/2}|\Lambda|^{1/2}} \cdot e^{-\frac{1}{2}(\bar{x}-\bar{m})\Lambda^{-1}(\bar{x}-\bar{m})^T}$$

$$\bar{m} = E\{\bar{\mathbf{X}}\} \qquad \Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1N} \\ \vdots & \ddots & \vdots \\ \lambda_{N1} & \cdots & \lambda_{NN} \end{pmatrix} \qquad \lambda_{ij} = Cov\{\mathbf{X}_i, \mathbf{X}_j\}$$

If $\mathbf{X}_1,\dots,\mathbf{X}_N$ *are pairwise uncorrelated.* $\Rightarrow \Lambda = \begin{pmatrix} \sigma^2_{\mathbf{X}_1} & & 0 \\ & \ddots & \\ 0 & & \sigma^2_{\mathbf{X}_N} \end{pmatrix} \Rightarrow$

$$\Rightarrow f_{\bar{\mathbf{X}}}(\bar{x}) = \frac{1}{(2\pi)^{N/2} \cdot \prod_i \sigma_{\mathbf{X}_i}} \cdot e^{-\frac{1}{2}\sum_i \frac{(x_i - m_{\mathbf{X}_i})^2}{\sigma^2_{\mathbf{X}_i}}} = \prod_{i=1}^{N} f_{\mathbf{X}_i}(x_i)$$

$\therefore$ *Independent*

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