

TSDT14 Signal Theory

Lecture 2

Stochastic processes

Mikael Olofsson
Department of EE (ISY)
Div. of Communication Systems

Multi-Dimensional Stochastic Variables

Joint Probability Distribution:

$$F_{X_1, \dots, X_N}(x_1, \dots, x_N) = \Pr\{X_1 \leq x_1, \dots, X_N \leq x_N\}$$

Joint Probability Density:

$$f_{X_1, \dots, X_N}(x_1, \dots, x_N) = \frac{\partial^N}{\partial x_1 \dots \partial x_N} F_{X_1, \dots, X_N}(x_1, \dots, x_N)$$

Vector notation:

$$\bar{X} = (X_1, \dots, X_N), \quad \bar{x} = (x_1, \dots, x_N), \quad F_{\bar{X}}(\bar{x}), \quad f_{\bar{X}}(\bar{x})$$

(Non-)dependencies of Multi-Dimensional Variables

Mutually Independent:

$$F_{\bar{X}}(\bar{x}) = \prod_{i=1}^N F_{X_i}(x_i)$$



Pairwise Independent:

The variables X_i and X_j are independent for all $i \neq j$.



Mutually Uncorrelated:

$$E\left\{\prod_{i \in I} X_i\right\} = \prod_{i \in I} E\{X_i\}$$

For all $I \subset \{1, 2, \dots, N\}$



Pairwise Uncorrelated:

The variables X_i and X_j are uncorrelated for all $i \neq j$.

Jointly Gaussian Variables

$\bar{X} = (X_1, \dots, X_N)$ is called *Jointly Gaussian* if the following holds:

$$f_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^{N/2} |\Delta|^{1/2}} \cdot e^{-\frac{1}{2}(\bar{x} - \bar{m})^T \Delta^{-1} (\bar{x} - \bar{m})}$$

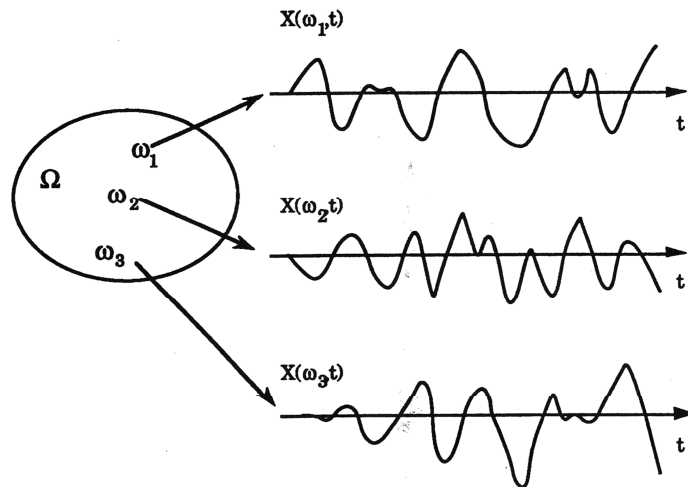
$$\bar{m} = E\{\bar{X}\} \quad \Delta = \begin{pmatrix} \lambda_{11} & \dots & \lambda_{1N} \\ \vdots & & \vdots \\ \lambda_{N1} & \dots & \lambda_{NN} \end{pmatrix} \quad \lambda_{ij} = \text{Cov}\{X_i, X_j\}$$

If $X_1, \dots, X_N$ are pairwise uncorrelated. $\Rightarrow \Delta = \begin{pmatrix} \sigma_{X_1}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{X_N}^2 \end{pmatrix} \Rightarrow$

$$\Rightarrow f_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^{N/2} \prod_i \sigma_{X_i}} \cdot e^{-\frac{1}{2} \sum_i \frac{(x_i - m_{X_i})^2}{\sigma_{X_i}^2}} = \prod_{i=1}^N f_{X_i}(x_i)$$

$\therefore$ Independent

Stochastic Process



Examples of Stochastic Processes

Ex 1: Finite number of realizations – This one is EPP:

$$X(t) = \sin(t + \Phi), \quad \Phi \in \{0, \pi/2, \pi, 3\pi/2\}$$

Ex 2: Infinite number of realizations – This one is EPP:

$$X(t) = A \cdot \sin(t), \quad A \sim N(0, 1)$$

Exactly Predictable Process

Definition:

A process is said to be exactly predictable if there exists a finite interval

$$t_1 \leq t \leq t_2$$

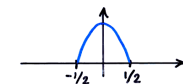
such that it is enough to know a realization in this interval to know the whole realization.

Examples of Stochastic Processes cont'd

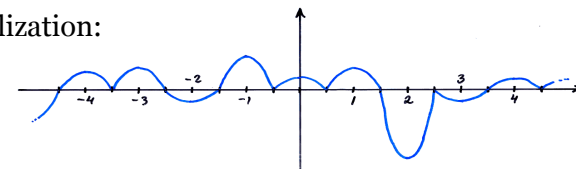
Ex 3: Infinite number of realizations – This one is not EPP:

$$X(t) = \sum A_k p(t - k), \quad p(t) = \begin{cases} \cos(\pi t), & |t| < 1/2 \\ 0, & \text{elsewhere} \end{cases}$$

$$\{A_k\} \text{ independent, } N(0, 1)$$



A realization:



Distributions and Densities

One time instance

Distribution: $F_{\mathbf{X}(t)}(x) = \Pr\{\mathbf{X}(t) \leq x\}$

Density: $f_{\mathbf{X}(t)}(x) = \frac{d}{dx} F_{\mathbf{X}(t)}(x)$

Two time instances

Distribution: $F_{\mathbf{X}(t_1), \mathbf{X}(t_2)}(x_1, x_2) = \Pr\{\mathbf{X}(t_1) \leq x_1, \mathbf{X}(t_2) \leq x_2\}$

Density: $f_{\mathbf{X}(t_1), \mathbf{X}(t_2)}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{\mathbf{X}(t_1), \mathbf{X}(t_2)}(x_1, x_2)$

Multiple Time Instances

Vector notation: $\bar{t} = (t_1, t_2, \dots, t_N)$
 $\mathbf{X}(\bar{t}) = (\mathbf{X}(t_1), \mathbf{X}(t_2), \dots, \mathbf{X}(t_N))$
 $\bar{x} = (x_1, x_2, \dots, x_N)$

Distribution: $F_{\mathbf{X}(\bar{t})}(\bar{x}) = \Pr\{\mathbf{X}(t_1) \leq x_1, \mathbf{X}(t_2) \leq x_2, \dots, \mathbf{X}(t_N) \leq x_N\}$

Density: $f_{\mathbf{X}(\bar{t})}(\bar{x}) = \frac{\partial^N}{\partial x_1 \partial x_2 \dots \partial x_N} F_{\mathbf{X}(\bar{t})}(\bar{x})$

Examples of Distributions and Densities

$\mathbf{X}(t) = A \cdot \sin(t)$, A is a stochastic variable

$F_{\mathbf{X}(t)}(x) = \Pr\{\mathbf{X}(t) \leq x\} = \Pr\{A \cdot \sin(t) \leq x\}$

$$= \begin{cases} \Pr\{A \leq \frac{x}{\sin(t)}\} = F_A(\frac{x}{\sin(t)}), & t: \sin(t) > 0 \\ \Pr\{0 \leq x\} = u(x), & t: \sin(t) = 0 \\ \Pr\{A \geq \frac{x}{\sin(t)}\} = 1 - F_A(\frac{x}{\sin(t)}), & t: \sin(t) < 0 \end{cases}$$

$$f_{\mathbf{X}(t)}(x) = \frac{d}{dx} F_{\mathbf{X}(t)}(x) = \begin{cases} \frac{1}{\sin(t)} \cdot f_A(\frac{x}{\sin(t)}), & t: \sin(t) > 0 \\ \delta(x), & t: \sin(t) = 0 \\ -\frac{1}{\sin(t)} \cdot f_A(\frac{x}{\sin(t)}), & t: \sin(t) < 0 \end{cases}$$

$$= \begin{cases} \delta(x), & t = k \cdot \pi, \quad k \in \mathbb{Z} \\ \frac{1}{|\sin(t)|} \cdot f_A(\frac{x}{\sin(t)}), & \text{elsewhere} \end{cases}$$

Ensemble Averages

Mean: $m_{\mathbf{X}}(t) = E\{\mathbf{X}(t)\}$
 $= \int_{-\infty}^{\infty} x f_{\mathbf{X}(t)}(x) dx$

Quadratic mean: $E\{\mathbf{X}^2(t)\} = \int_{-\infty}^{\infty} x^2 f_{\mathbf{X}(t)}(x) dx$

Variance: $\sigma_{\mathbf{X}}^2(t) = E\{(\mathbf{X}(t) - m_{\mathbf{X}}(t))^2\}$
 $= E\{\mathbf{X}^2(t)\} - m_{\mathbf{X}}^2(t)$

Std deviation: $\sigma_{\mathbf{X}}(t)$

Functions of time!

Auto-correlation function (ACF):

$$r_{\mathbf{X}}(t_1, t_2) = E\{\mathbf{X}(t_1) \mathbf{X}(t_2)\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{\mathbf{X}(t_1), \mathbf{X}(t_2)}(x_1, x_2) dx_1 dx_2$$

Symmetry: $r_{\mathbf{X}}(t_1, t_2) = r_{\mathbf{X}}(t_2, t_1)$

Power: $r_{\mathbf{X}}(t, t) = E\{\mathbf{X}^2(t)\}$

Special case: $\mathbf{X}(t)$ fun of stock var A :

$$\mathbf{X}(t) = g(t, A)$$

$$r_{\mathbf{X}}(t_1, t_2) = \int_{-\infty}^{\infty} g(t_1, a) g(t_2, a) f_A(a) da$$

Example of Ensemble Averages

$X(t) = A \sin(t)$, A is a stochastic variable

$$m_X(t) = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx = \int_{-\infty}^{\infty} a \sin(t) \cdot f_A(a) da = \sin(t) \cdot \int_{-\infty}^{\infty} a f_A(a) da = \sin(t) \cdot m_A$$

$$E\{X^2(t)\} = \int_{-\infty}^{\infty} x^2 f_{X(t)}(x) dx = \int_{-\infty}^{\infty} (a \sin(t))^2 \cdot f_A(a) da = \sin^2(t) \int_{-\infty}^{\infty} a^2 f_A(a) da = \sin^2(t) \cdot E\{A^2\}$$

$$\sigma_X^2(t) = E\{X^2(t)\} - m_X^2(t) = \sin^2(t) (E\{A^2\} - m_A^2) = \sin^2(t) \cdot \sigma_A^2$$

$$\begin{aligned} r_X(t_1, t_2) &= \int_{-\infty}^{\infty} a \sin(t_1) \cdot a \sin(t_2) \cdot f_A(a) da \\ &= \sin(t_1) \cdot \sin(t_2) \cdot \int_{-\infty}^{\infty} a^2 f_A(a) da \\ &= \sin(t_1) \cdot \sin(t_2) \cdot E\{A^2\} \end{aligned}$$

Auto-Correlation and Auto-Covariance

Auto-correlation (ACF):

$$r_X(t_1, t_2) \triangleq E\{X(t_1)X(t_2)\} = \iint_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)}(x_1, x_2) dx_1 dx_2,$$

$$r_X(t_1, t_2) = r_X(t_2, t_1)$$

Auto-covariance (related concept):

$$\lambda_X(t_1, t_2) \triangleq E\{(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))\}$$

$$\lambda_X(t_1, t_2) = r_X(t_1, t_2) - m_X(t_1)m_X(t_2).$$

Stationarity 1(2)

Stationarity is statistical invariance to a shift of the time origin.

Definition:

Consider time instances $\bar{t} = (t_1, \dots, t_N)$ and shifted time instances $\bar{u} = \bar{t} + \Delta = (t_1 + \Delta, \dots, t_N + \Delta)$. The process $X(t)$ is said to be stationary in the strict sense (SSS) if

$$F_{X(\bar{t})}(\bar{x}) = F_{X(\bar{u})}(\bar{x})$$

holds for all N and all choices of $\bar{t}$ and Δ .

Equivalence:

$$F_{X(\bar{t})}(\bar{x}) = F_{X(\bar{u})}(\bar{x}) \Leftrightarrow f_{X(\bar{t})}(\bar{x}) = f_{X(\bar{u})}(\bar{x})$$

Stationarity 2(2)

Mean: $m_X(t) = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx \stackrel{\text{stat.}}{=} \int_{-\infty}^{\infty} x f_{X(t+\Delta)}(x) dx = m(t+\Delta) \quad \forall \Delta$
Thus Constant.

ACF: $r_X(t_1, t_2) = \iint_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)}(x_1, x_2) dx_1 dx_2$
 $= \iint_{-\infty}^{\infty} x_1 x_2 f_{X(t_1+\Delta), X(t_2+\Delta)}(x_1, x_2) dx_1 dx_2$
 $= r_X(t_1+\Delta, t_2+\Delta)$ Dep. on t_1, t_2

Notation: m_X and $r_X(\tau)$ $\tau = t_1 - t_2$

Wide sense stationarity, definition:

If the above holds, then the process $X(t)$ is said to be stationary in the wide sense (WSS).

Gaussian Processes

Recall: $\bar{X} = (X_1, \dots, X_N)$ is called Jointly Gaussian if the following holds:

$$f_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^{N/2} |\Delta|^{1/2}} \cdot e^{-\frac{1}{2}(\bar{x} - \bar{m}) \Delta^{-1} (\bar{x} - \bar{m})^T}$$

$$\bar{m} = E\{\bar{X}\} \quad \Delta = \begin{pmatrix} \lambda_{11} & \dots & \lambda_{1N} \\ \vdots & \ddots & \vdots \\ \lambda_{N1} & \dots & \lambda_{NN} \end{pmatrix} \quad \lambda_{ij} = \text{Cov}\{X_i, X_j\}$$

Definition: A stochastic process is called *Gaussian* if all its multidimensional PDFs correspond to jointly Gaussian variables.

Theorem: A Gaussian process that is stationary in the wide sense is also stationary in the strict sense.

Power-Spectral Density (PSD)

Definition: Fourier transform of the ACF:

$$R_X(f) = \mathcal{F}\{r_X(\tau)\} = \int_{-\infty}^{\infty} r_X(\tau) e^{-j2\pi f\tau} d\tau$$

Inverse:

$$r_X(\tau) = \mathcal{F}^{-1}\{R_X(f)\} = \int_{-\infty}^{\infty} R_X(f) e^{j2\pi f\tau} df$$

Power:

$$E\{X^2(t)\} = r_X(0) = \int_{-\infty}^{\infty} R_X(f) df$$

Ergodicity 1(2)

A WSS process:

$$m_X = E\{X(t)\}$$

Time-average of one realization:

$$m_T = \frac{1}{2T} \int_{-T}^T x(t) dt$$

Time-average of the process:

$$M_T = \frac{1}{2T} \int_{-T}^T X(t) dt$$

$$E\{M_T\} = E\left\{\frac{1}{2T} \int_{-T}^T X(t) dt\right\} = \frac{1}{2T} \int_{-T}^T E\{X(t)\} dt = \frac{1}{2T} \int_{-T}^T m_X dt = m_X$$

Definition: If $\lim_{T \rightarrow \infty} E\{(M_T - m_X)^2\} = 0$ then $X(t)$ is said to be ergodic with respect to the mean, and we write

$$m_X = \text{l.i.m.}_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) dt \quad (\text{Limes in mean square})$$

Interpretation: The time-average of a process is very close to the ensemble mean with probability that is very close to 1 ($\rightarrow 1, T \rightarrow \infty$).

Ergodicity 2(2)

Definition: A process that is ergodic with respect to all ensemble averages is simply said to be ergodic.

Theorem: An ergodic process is SSS.

Theorem: A SSS Gaussian process with mean zero is ergodic if and only if its PSD has no impulses.

Theorem: A SSS process, $X(t)$, is ergodic with respect to its mean if and only if

$$\text{l.i.m.}_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T r_X(\tau) d\tau = m_X^2$$

holds.

Mikael Olofsson
ISY/CommSys

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