TSKS01 Digital Communication
Lecture 4

Digital Modulation – Basis Functions and Basic Signal Detection

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Last Time – Digital Modulation

Signals: $s_i(t) = \sum_{j=0}^{N-1} s_{i,j}\phi_j(t), \quad i = 0,1,\dots,M-1, \quad 0 \le t < T$

ON basis

AWGN: $W(t) = \sum_{j=0}^{N-1} W_j\phi_j(t) + W'(t)$ $\quad W_j = (W, \phi_j)$

Irrelevant

Gaussian with mean 0.

Received: $X(t) = s_i(t) + W(t) = \sum_{j=0}^{N-1} X_j\phi_j(t) + W'(t)$ $\quad X_j = (X, \phi_j)$

Gaussian with mean $s_{i,j}$.

Vectors: $\overline{X} = \overline{s}_i + \overline{W}$

$$\begin{pmatrix} X_0 \\ \vdots \\ X_{N-1} \end{pmatrix} \quad \begin{pmatrix} s_{i,0} \\ \vdots \\ s_{i,N-1} \end{pmatrix} \quad \begin{pmatrix} W_0 \\ \vdots \\ W_{N-1} \end{pmatrix}$$

Orthogonal noise components are statistically independent.

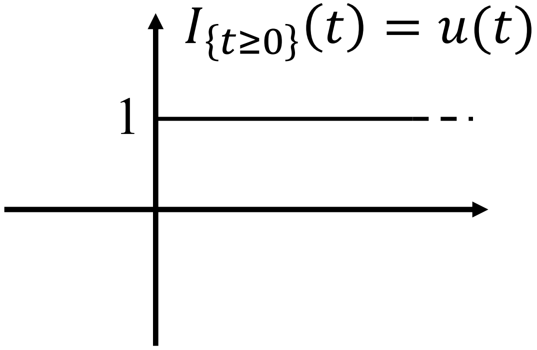
$\sigma_{W_i}^2 = \sigma_{X_i}^2 = R_W(f) = N_0/2$

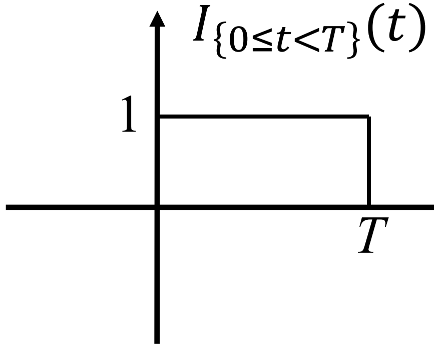
Indicator Function

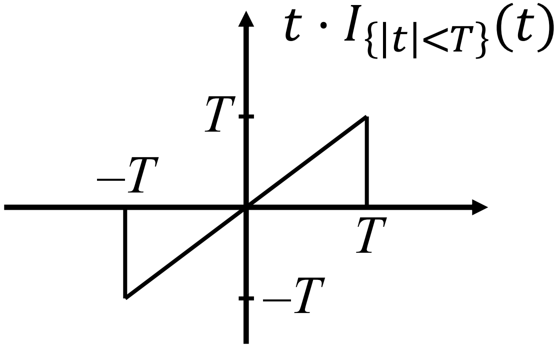
Definition: The *indicator function* $I_A(t)$ of the set A is given by

$$I_A(t) = \begin{cases} 1, & t \in A \\ 0, & \text{elsewhere} \end{cases}$$

Examples:







Example of Basis Functions ($N = 1$)

- Time-limited baseband signal

$$\tilde{\phi}_0(t) = \sqrt{\frac{1}{T}} I_{\{0 \le t < T\}}(t)$$

- Time-limited passband signal (carrier frequency f_c)

$$\phi_0(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \, I_{\{0 \le t < T\}}(t)$$

  - $\|\phi_0\|^2 = \frac{2}{T} \int_0^T \cos^2(2\pi f_c t) \, dt = 1 - \frac{2}{T} \int_0^T \cos(4\pi f_c t) \, dt$

Equal to one if $4 f_c T$ is an integer

Example of Basis Functions ($N = 2$)

- Time-limited passband signal (carrier frequency f_c)

$$\phi_0(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)\, I_{\{0 \le t < T\}}(t)$$

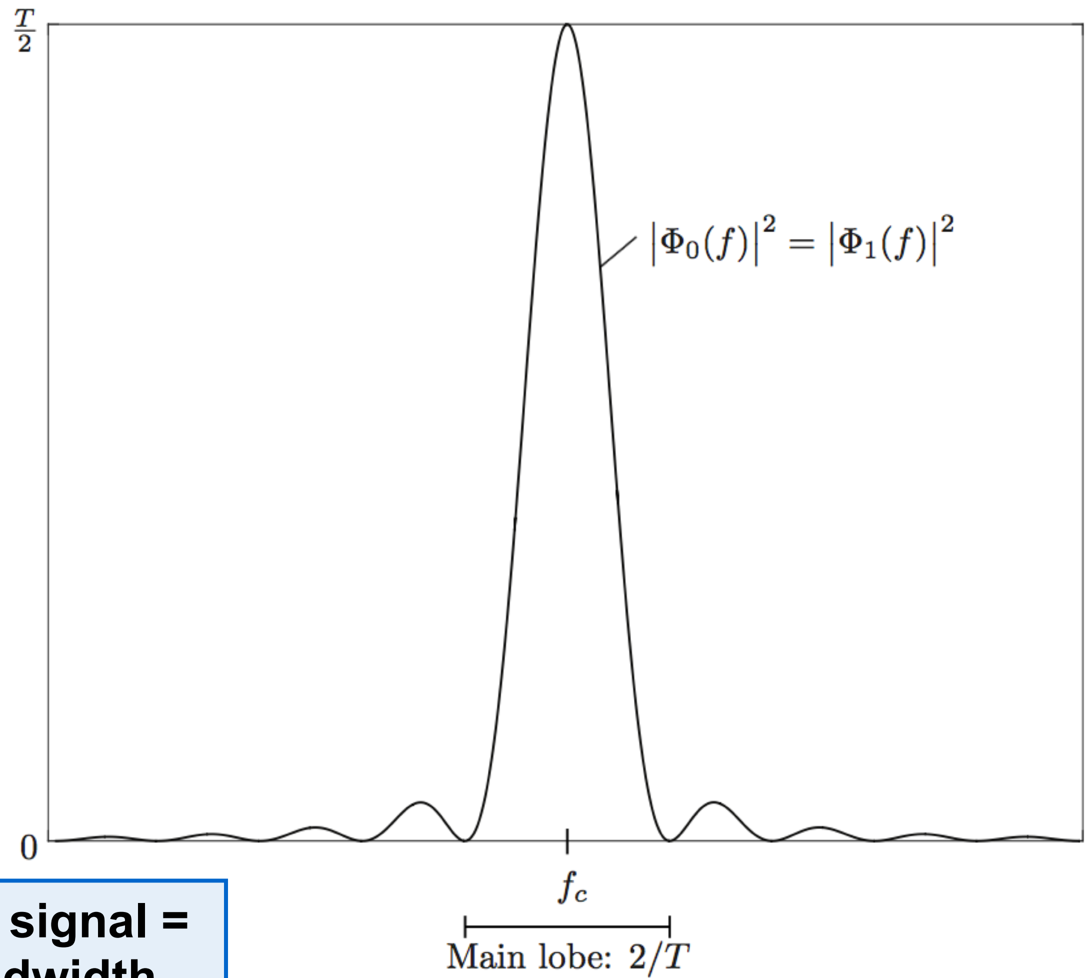
$$\phi_1(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t)\, I_{\{0 \le t < T\}}(t)$$

- When are these functions orthogonal?

$$\int_{-\infty}^{\infty} \phi_0(t)\phi_1(t) dt = \cdots = -\frac{1}{T} \int_0^T \sin(4\pi f_c t)\, dt$$

Equal to zero if $2 f_c T$ is an integer

Energy Spectra: $|\Phi_0(f)|$ and $|\Phi_1(f)|$



Time-limited signal = Infinite bandwidth

Approximation

Count main lobe: $2/T$

Example of Basis Functions ($N > 2$)

- How to construct more than two basis functions?
  - One can show that $\cos(2\pi f_0 t)\, I_{\{0\le t<T\}}(t)$ and $\cos(2\pi f_k t)\, I_{\{0\le t<T\}}(t)$ are orthogonal if $(f_0 - f_k)T$ is an integer

- Proposal ($N/2$ is an integer):

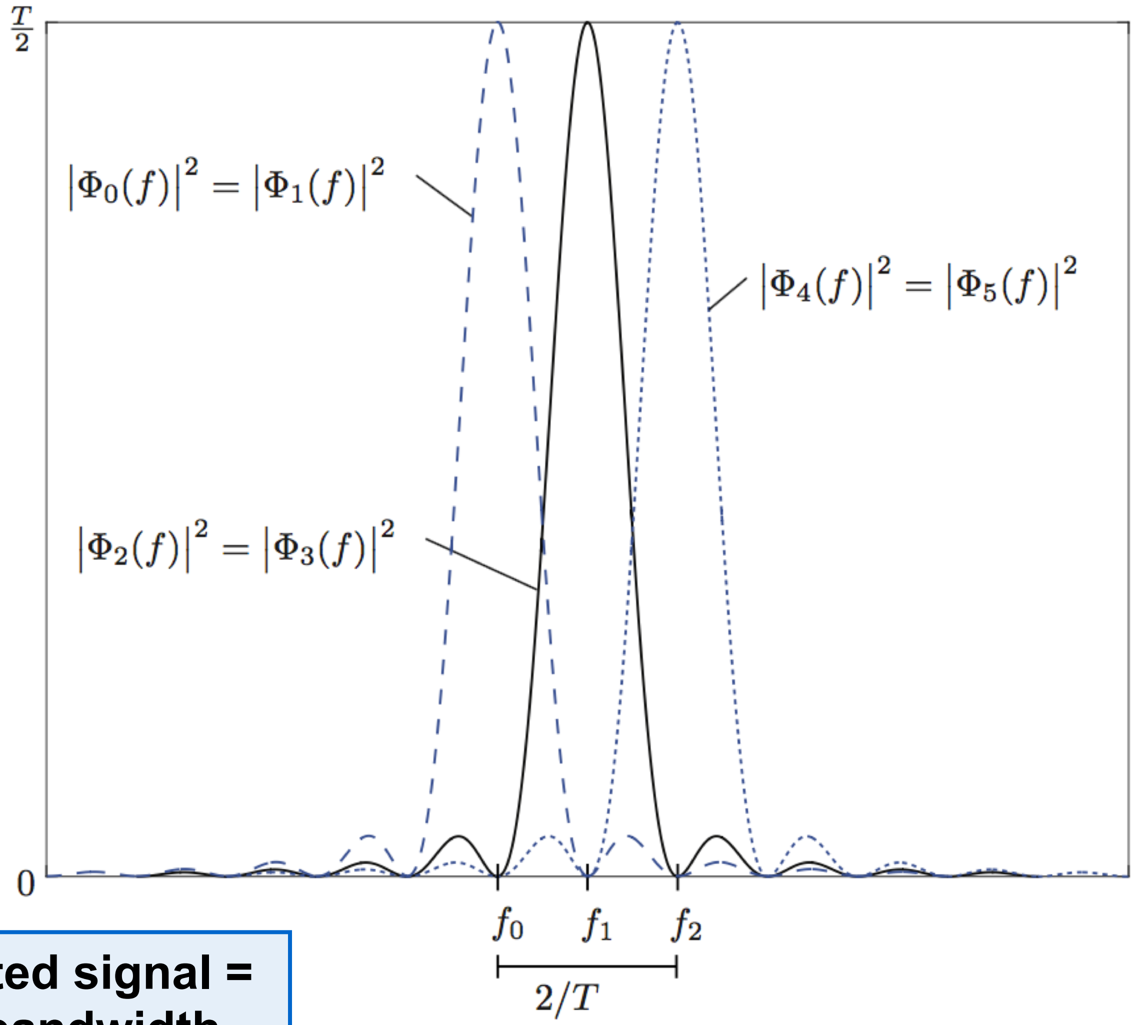
$$f_k = f_0 + \frac{k}{T} \qquad \text{for } k = 0,1,\ldots,\frac{N}{2}-1$$

$$\phi_{2k}(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_k t)\, I_{\{0\le t<T\}}(t)$$

$$\phi_{2k+1}(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_k t)\, I_{\{0\le t<T\}}(t)$$

$N/2$ pairs of cosine and sine

Energy Spectra: $|\Phi_n(f)|$ $\quad n = 0, \ldots N-1$



Time-limited signal = Infinite bandwidth

Approximation

Count main lobes:

$$\left(\frac{N}{2} + 1\right) \cdot \frac{1}{T}$$

Single and Multi Carrier Modulation 1(2)

- Given total bandwidth B
  - Approximate bandwidth: count only the main lobe

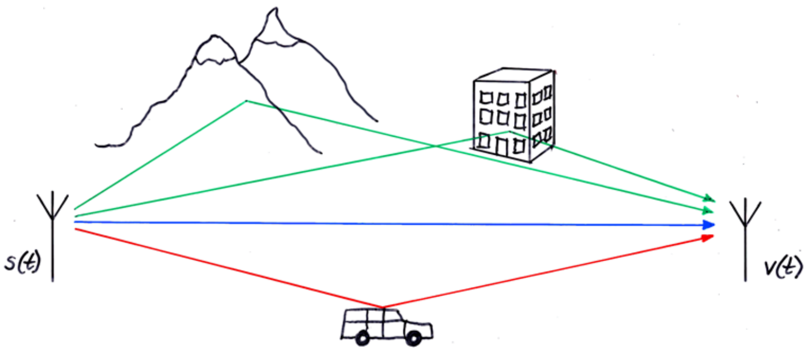
- Single Carrier ($N = 2$)
  - $B = \frac{2}{T} \qquad \leftrightarrow \qquad T = \frac{2}{B}$

- Multi Carrier ($N > 2$, $N/2$ integer)
  - $B = \left(\frac{N}{2} + 1\right) \cdot \frac{1}{T} \qquad \leftrightarrow \qquad T = \left(\frac{N}{2} + 1\right) \cdot \frac{1}{B}$

- Example: $N = 1024$: T is 256.5 times larger with multi carrier
  - Main lobe is 256.6 times smaller

Single and Multi Carrier Modulation 2(2)

- Frequency response of channel
  - Changes with frequency
  - Approx. fixed over small interval



$$h(t) = \delta(t-\tau_1) + \delta(t-\tau_2) + \delta(t-\tau_3) + \delta(t-\tau_4)$$

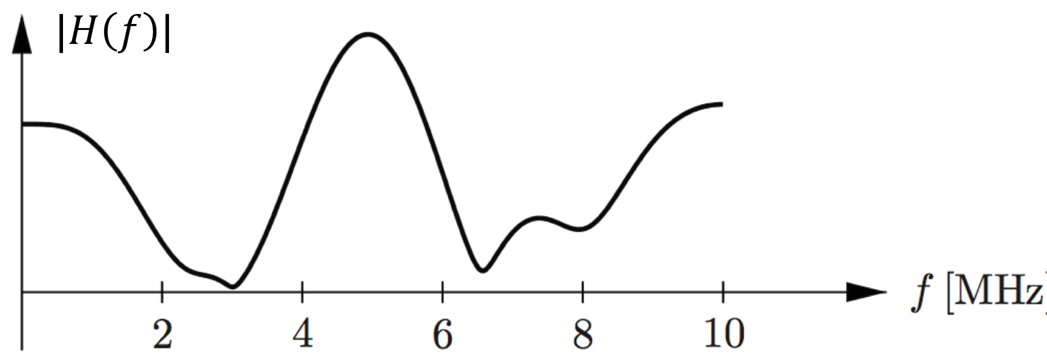
Multi Carrier

Select N large enough:

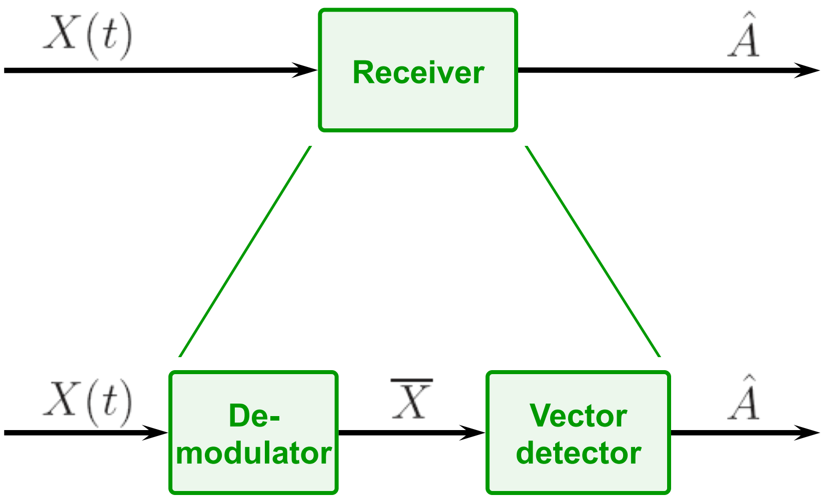
Make $H(f)$ approximately constant over the main lobe $2/T$

Can ignore channel:

This is what we will do right now



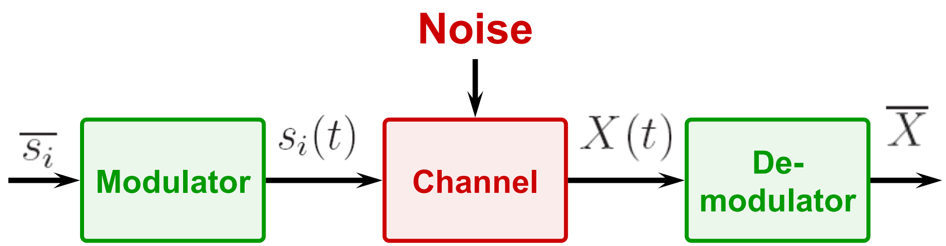
Receiving a Modulated Signal



$$X_j \triangleq (X, \phi_j)$$

Next topic: The demodulator and vector detector

Demodulation



Projections:

$$s_{i,j} = (s_i, \phi_j) = \int_0^T s_i(t)\,\phi_j(t)\,dt,$$

$$W_j = (W, \phi_j) = \int_0^T W(t)\,\phi_j(t)\,dt$$

$$X_j = (X, \phi_j) = \int_0^T X(t)\,\phi_j(t)\,dt$$

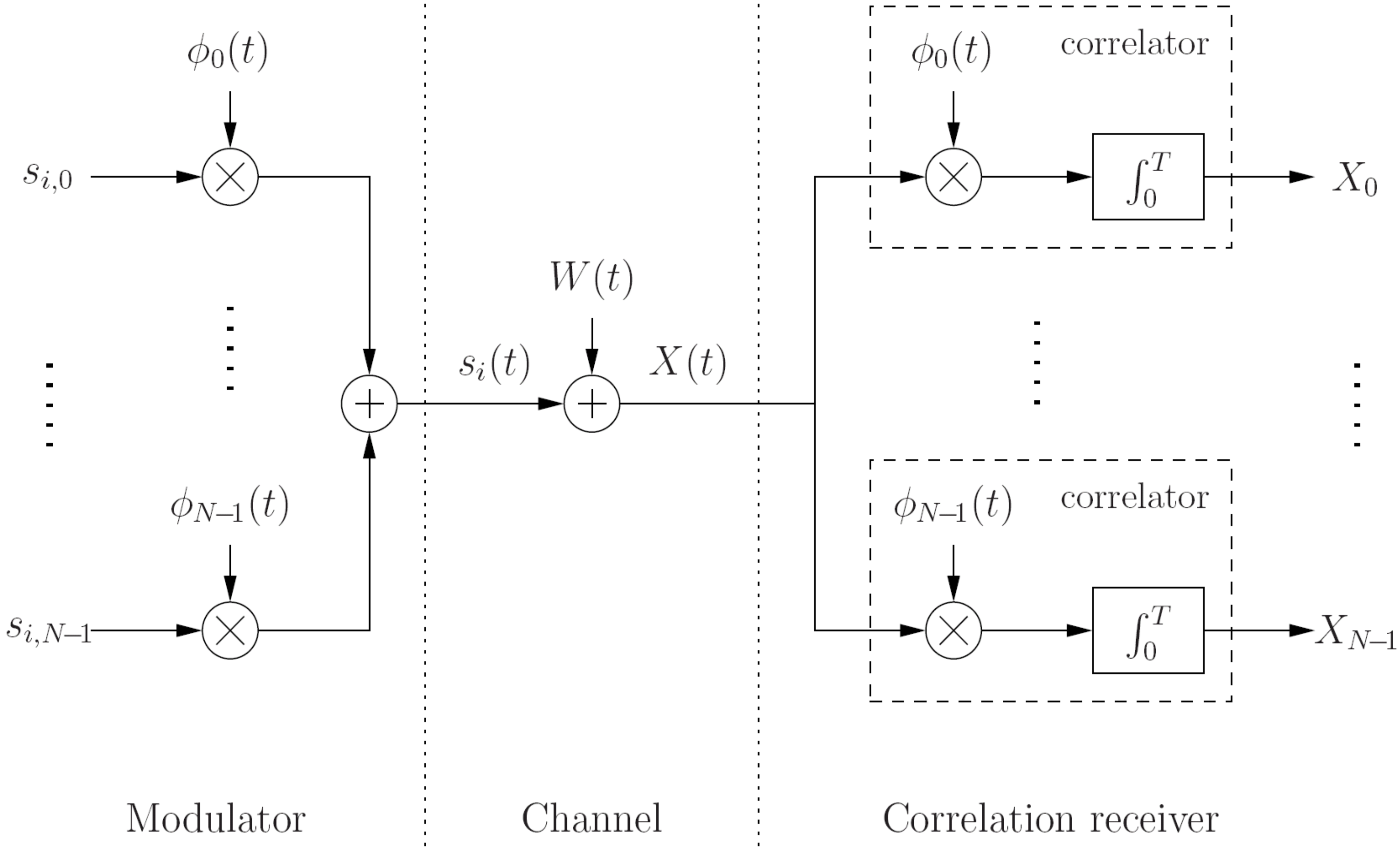
Do this!

$$X_j = s_{i,j} + W_j \qquad j \in \{0, 1, \ldots, N-1\}.$$

Signals are projected on basis functions

Noise is projected on basis functions

Correlation Receiver

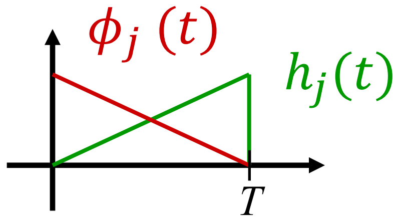


Matched Filters

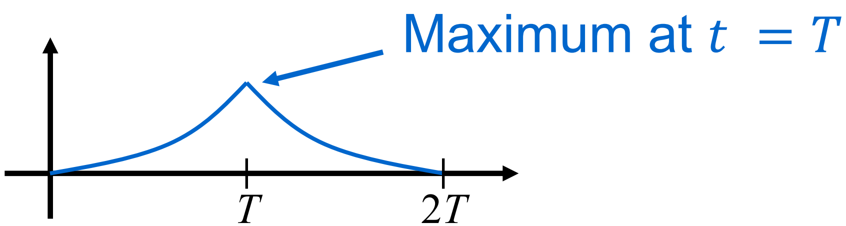
Definition:

A filter with impulse response $h_j(t) = \phi_j(T - t)$ is matched to $\phi_j(t)$

Example:



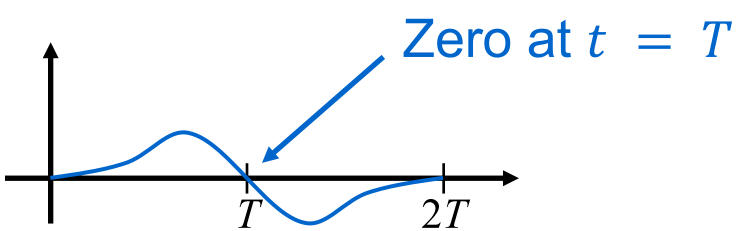
Usage $(\phi_j * h_j)(t)$:



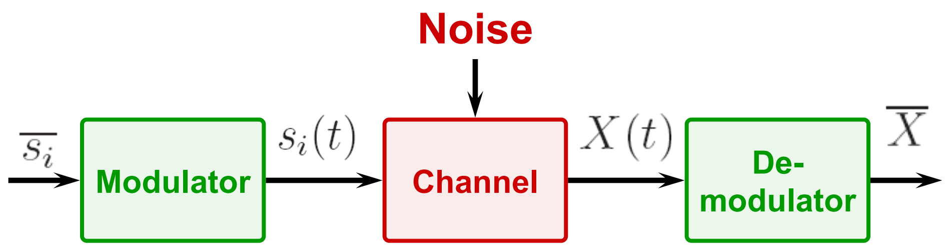
Orthogonal signal $\phi_i(t)$:

$$(\phi_i(t), \phi_j(t)) = 0$$

Then $(\phi_i * h_j)(t)$:



Demodulation Using Matched Filters



Filter: $h_j(t) \triangleq \phi_j(T-t)$ Matched to $\phi_j(t)$

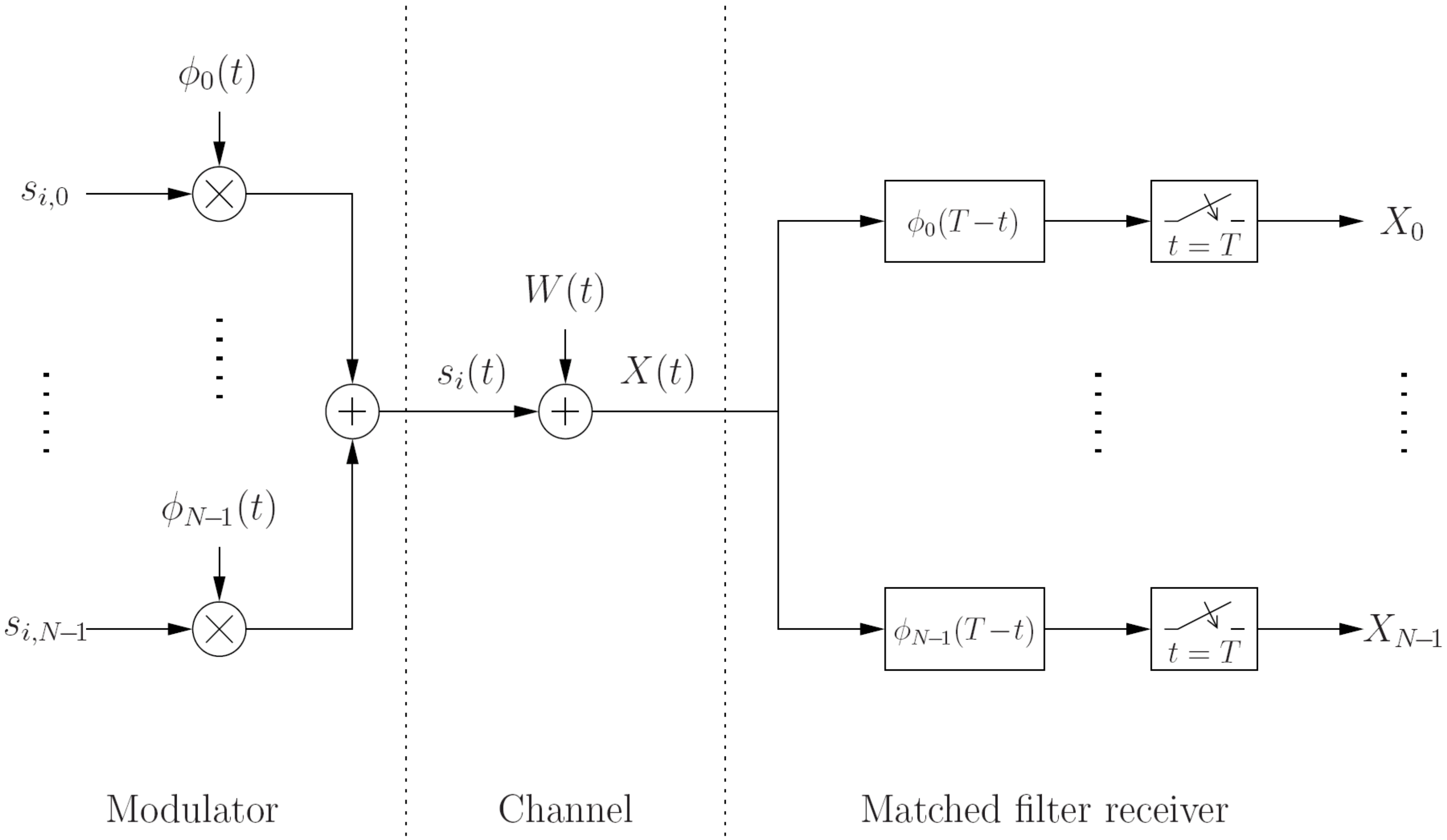
Output: $Y_j(t) = (X * h_j)(t)$ ← Do this!

$$= \int_{-\infty}^{\infty} X(\tau)\, h_j(t-\tau)\, d\tau = \int_{t-T}^{t} X(\tau)\, \phi_j(T-t+\tau)\, d\tau$$

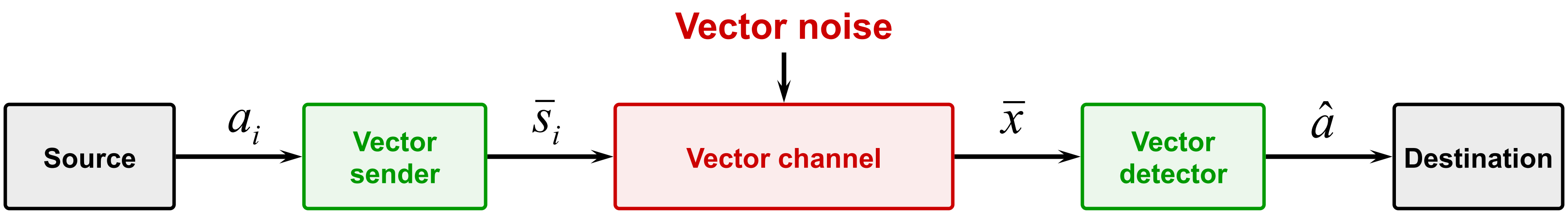
Sample at $t = T$: ← And this!

$$Y_j(T) = \int_{0}^{T} X(\tau)\, \phi_j(\tau)\, d\tau = (X, \phi_j) = X_j$$

Matched Filter Receiver



The Vector Detector



The task of the vector detector:

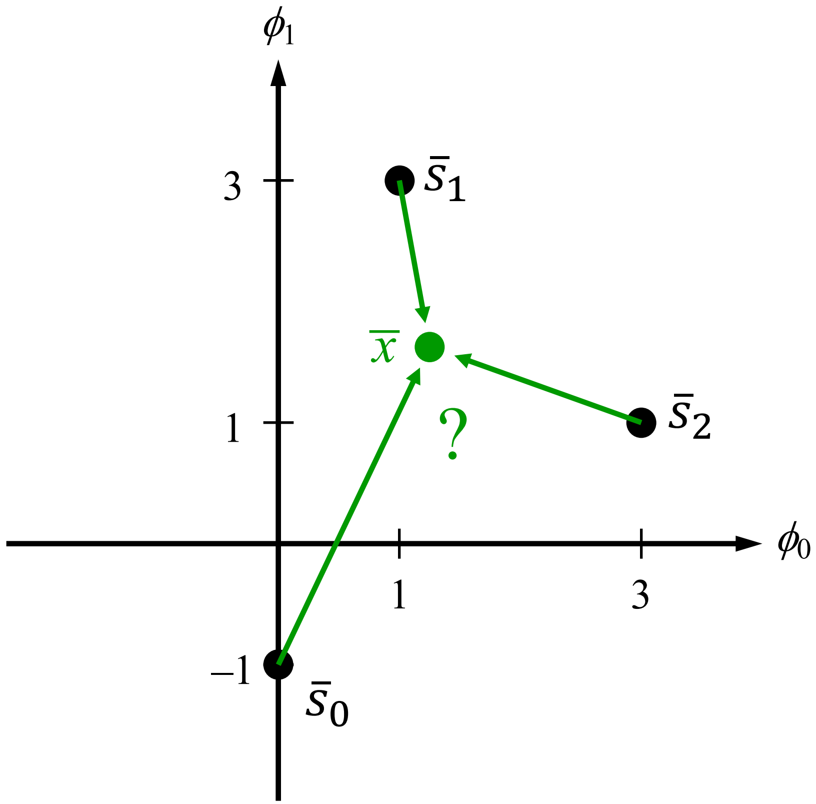
Observe $\bar{x}$ and output $\hat{a}$ according to a well chosen decision rule, designed to minimize the error probability.

What do we know?

All entities are realizations of stochastic variables.

Use statistical descriptions of those stochastic variables!

Vector Detection



Received vector: $\bar{x}$

Goal: Minimize the error probability

$$\Pr\{A \neq \hat{A} | \bar{X} = \bar{x}\}$$

Sent symbol

Estimated symbol

Received vector

Observed realization

Equivalent decision rule:

Set $\hat{a} = a_i$ if $\Pr\{A = a_k | \bar{X} = \bar{x}\}$ is maximized for $k = i$.

Detection

Bayes' rule: $$\Pr\{A = a_k | \bar{X} = \bar{x}\} = \frac{f_{\bar{X}|A}(\bar{x}|a_k)}{f_{\bar{X}}(\bar{x})} \Pr\{A = a_k\}$$

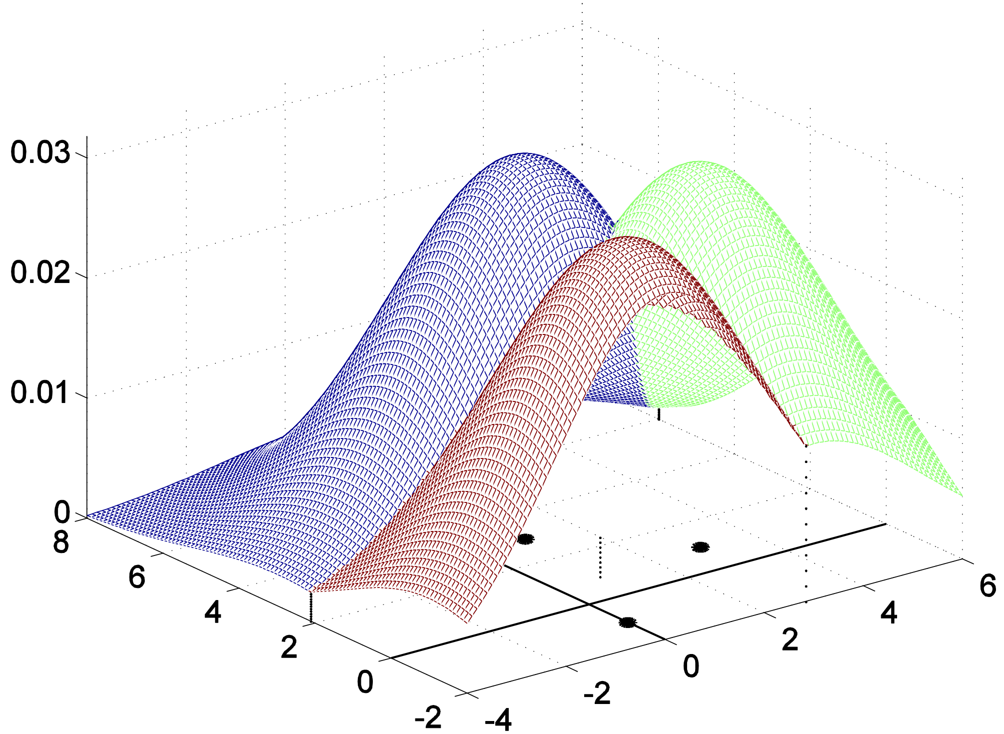
Independent of a_k

New decision rule:

Set $\hat{a} = a_i$ if

$f_{\bar{X}|A}(\bar{x}|a_k) \Pr\{A = a_k\}$

is maximized for $k = i$.



Maximum Likelihood (ML) Detection

Assumption: $\Pr\{A = a_i\} = \frac{1}{M}$ for $i = 0,1,\dots,M-1$

ML decision rule: Set $\hat{a} = a_i$ if $f_{\bar{X}|A}(\bar{x}|a_k)$ is maximized for $k = i$.

Independent variables:

$$f_{\bar{X}|A}(\bar{x} \mid a_k) = \prod_{j=0}^{N-1} f_{X_j|A}(x_j \mid a_k) = \prod_{j=0}^{N-1} \frac{1}{\sqrt{\pi N_0}} \cdot e^{-(x_j - s_{k,j})^2/N_0} = (\pi N_0)^{-N/2} \cdot e^{-\frac{1}{N_0}\sum_{j=0}^{N-1}(x_j - s_{k,j})^2}$$

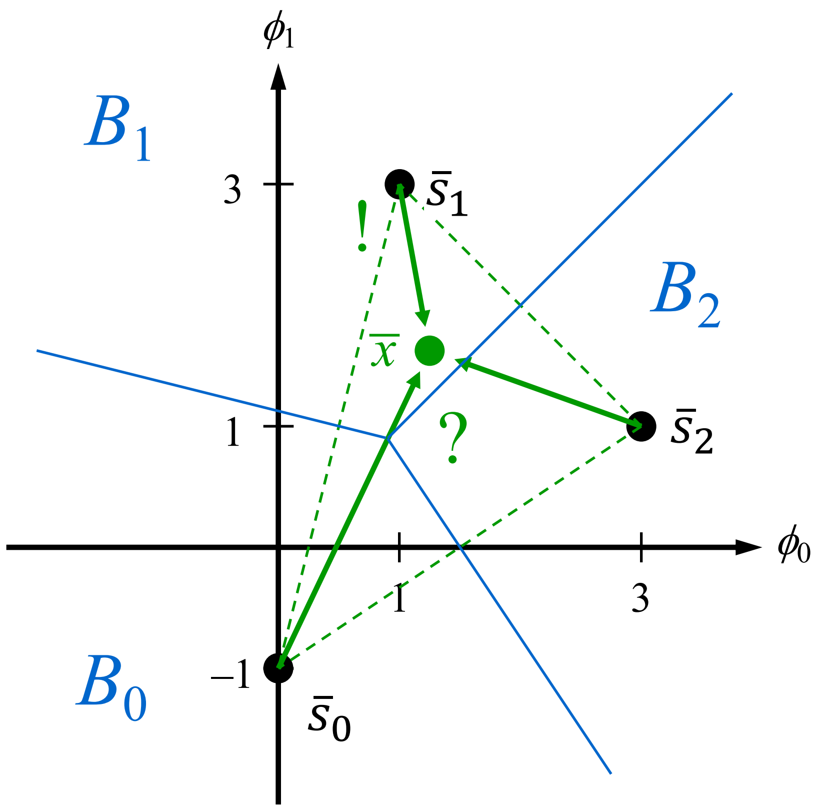
Natural logarithm (strictly increasing function):

Constants

$$\ln\left(f_{\bar{X}|A}(\bar{x} \mid a_k)\right) = -\frac{N}{2}\ln(\pi N_0) - \frac{1}{N_0}\sum_{j=0}^{N-1}(x_j - s_{k,j})^2 = -\frac{N}{2}\ln(\pi N_0) - \frac{1}{N_0} d^2(\bar{x}, \bar{s}_k)$$

Equivalent ML rule: Set $\hat{a} = a_i$ if $d(\bar{x}, \bar{s}_k)$ is minimized for $k = i$.

ML Decision Regions



Detect $\overline{x}$ as being the nearest signal.

Result:

Decision regions consist of all points closest to a signal point.

Notation:

B_i is the decision region of the signal vector $\bar{s}_i$.
Thus also of the signal $s_i(t)$ and of the message a_i.

Borders are orthogonal to straight lines between signals:

In 2 dimensions: Lines.
In 3 dimensions: Planes.
Higher dim: Hyperplanes.

Borders cut the lines mid-way.

Error Probability

Symbol error probability:

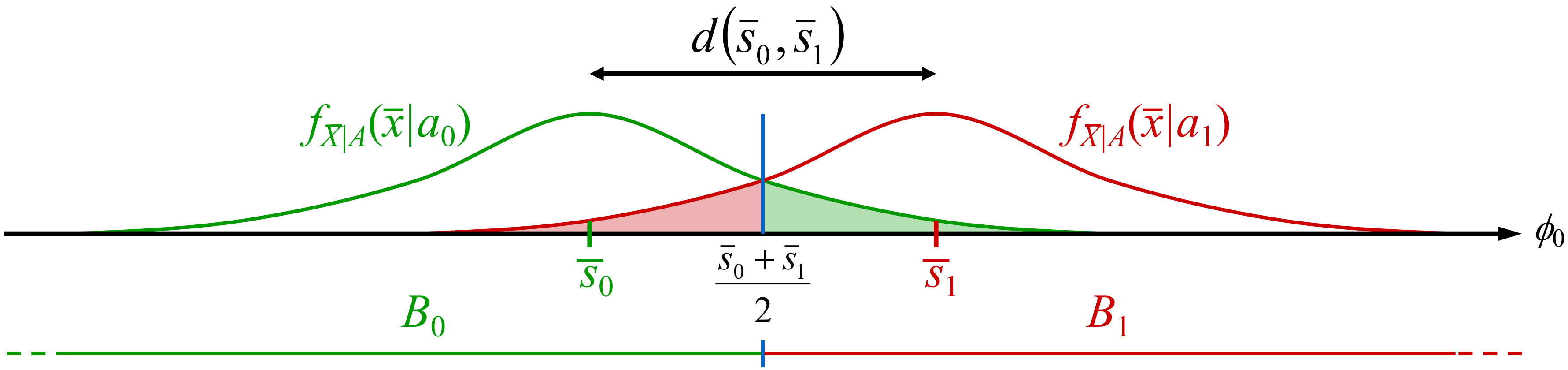
$$P_{\mathrm{e}} = \Pr\left\{\hat{A} \neq A\right\} = \sum_{i=0}^{M-1} \Pr\left\{A = a_i\right\} \cdot \Pr\left\{\hat{A} \neq a_i \mid A = a_i\right\}$$
$$= \sum_{i=0}^{M-1} \Pr\left\{A = a_i\right\} \cdot \Pr\left\{\overline{X} \notin B_i \mid A = a_i\right\}$$

ML detection: $\Pr\{A = a_i\} = \frac{1}{M}$ for $i = 0, 1, \ldots, M-1$:

$$P_{\mathrm{e}} = \frac{1}{M} \sum_{i=0}^{M-1} \Pr\left\{\overline{X} \notin B_i \mid A = a_i\right\} = \frac{1}{M} \sum_{i=0}^{M-1} \int \cdots \int_{\overline{x} \notin B_i} f_{\overline{X}|A}\left(\overline{x} \mid a_i\right) dx_0 \cdots dx_{N-1}$$

This is generally hard to calculate!

Special Case: Two signals in $N = 1$ Dimension



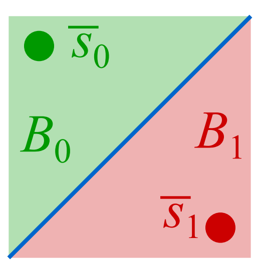
$$\Pr\left\{\bar{X} \notin B_0 \mid A = a_0\right\} = \Pr\left\{\bar{X} \in B_1 \mid A = a_0\right\} = \Pr\left\{ X_0 > \frac{s_{0,0} + s_{1,0}}{2} \,\middle|\, A = a_0 \right\}$$

$$= \Pr\left\{ W_0 > \frac{s_{0,0} + s_{1,0}}{2} - s_{0,0} \,\middle|\, A = a_0 \right\} = \Pr\left\{ W_0 > \frac{s_{1,0} - s_{0,0}}{2} \right\}$$

$$= \Pr\left\{ W_0 > \frac{d(s_0, s_1)}{2} \right\} = Q\left(\frac{d(s_0, s_1)/2}{\sqrt{N_0/2}} \right) = Q\left(\frac{d(s_0, s_1)}{\sqrt{2N_0}} \right)$$

Similarily for $\Pr\left\{\bar{X} \notin B_1 \mid A = a_1\right\}$. $\quad \Rightarrow \quad P_e = Q(\cdots)$

This works in all directions.



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