TSDT14 Signal Theory

Lecture 9
Reconstruction and Reconstruction Errors

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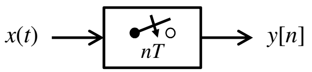
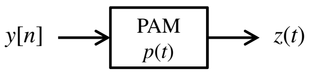
Sampling and PAM of Deterministic Signals – Summary



$$y[n] = x(nT)$$

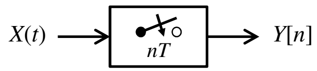
$$Y[\theta] = \frac{1}{T}\sum_m X\left(\frac{\theta - m}{T}\right)$$



$$z(t) = \sum_n y[n]\, p(t - kT)$$

$$Z(f) = P(f) Y[fT]$$

Sampling and PAM of WSS Processes – Summary

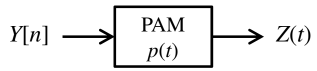


$$Y[n] = X(nT)$$

$$m_Y = m_X$$

$$r_Y[k] = r_X(kT)$$

$$R_Y[\theta] = \frac{1}{T}\sum_m R_X\left(\frac{\theta - m}{T}\right)$$



$$Z(t) = \sum_n Y[n]\, p(t - kT - \Psi)$$

$$m_Z = \frac{1}{T} P(0) m_Y$$

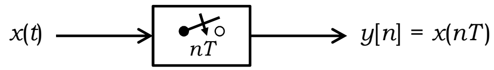
Ψ unif. $[0, T]$
Ψ & $Y[n]$ indep.

$$r_Z(\tau) = \frac{1}{T}\sum_k r_Y[k] (p * \tilde{p})(\tau - kT)$$

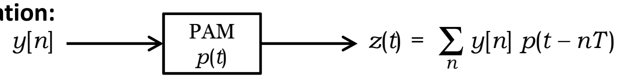
$$R_Z(f) = \frac{1}{T} |P(f)|^2 R_Y[fT]$$

Linear Mappings

Sampling:

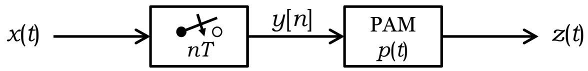


Pulse-Amplitude Modulation: (PAM)

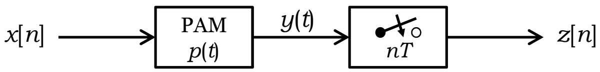


Reconstruction:

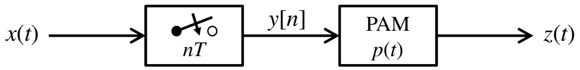
$t \to n \to t$:



$n \to t \to n$:



Sampling Theorem for Deterministic Signals



The Sampling Theorem:

Consider a signal $x(t)$, with spectrum $X(f)$ and $X(f) = 0$ for $|f| \geq f_0$. If $x(t)$ is sampled with sampling frequency f_s, then $x(t)$ can be reconstructed without error from the sampled signal if $f_s \geq 2f_0$ holds.

This means:

There exists a pulse shape $p(t)$, such that $x(t)$ can be written as

$$x(t) = \sum_n x(nT)\, p(t - nT)$$

if $f_s \geq 2f_0$ holds, where $f_s = 1/T$.

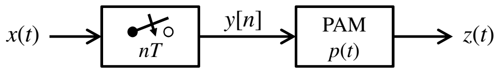
Fulfilled for:

Ideal reconstruction: $p(t) = \mathrm{sinc}(t/T)$

Reconstruction – Deterministic Case

$t \to n \to t:$

$$y[n] = x(nT) \qquad Y[\theta] = f_s \sum_m X\big((\theta - m) f_s\big)$$



$$z(t) = \sum_n y[n]\, p(t - nT) \qquad Z(f) = P(f)\, Y[f/f_s]$$

Total spectrum:

$$Z(f) = \; = f_s P(f) \sum_m X(f - m f_s)$$

Distorsion: $\varepsilon^2 = \int_{-\infty}^{\infty} \big(z(t) - x(t)\big)^2 dt \overset{\text{Parseval}}{=} \int_{-\infty}^{\infty} |Z(f) - X(f)|^2 df$

$$= \int_{-\infty}^{\infty} \Big| f_s P(f) \sum_m X(f - m f_s) - X(f) \Big|^2 df = \int_{-\infty}^{\infty} \Big| \big(f_s P(f) - 1\big) X(f) + f_s P(f) \sum_{m \neq 0} X(f - m f_s) \Big|^2 df$$

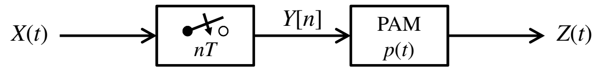
Ideal reconstruction:

$p(t) = \mathrm{sinc}(t/T)$

$P(f) = T\,\mathrm{rect}(fT)$

$$\varepsilon^2 = \underbrace{2 \int_{f_s/2}^{\infty} |X(f)|^2 df}_{\text{Bandlimiting distortion}} + \underbrace{2 \int_0^{f_s/2} \Big| \sum_{m \neq 0} X(f - m f_s) \Big|^2 df}_{\text{Aliasing distortion}}$$

The Sampling Theorem for Stochastic Processes



The Sampling Theorem:

Consider a process $X(t)$, with spectrum $R_X(f)$ and $R_X(f) = 0$ for $|f| \geq f_0$. If $X(t)$ is sampled with sampling frequency f_s, then $X(t)$ can be reconstructed without error from the sampled signal if $f_s \geq 2f_0$ holds.

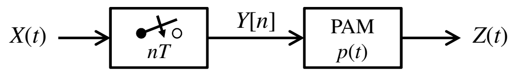
This means:

There exists a pulse shape $p(t)$, such that $\varepsilon^2 = \mathrm{E}\big\{ (Z(t) - X(t))^2 \big\} = 0$ holds.

Fulfilled for:

Ideal reconstruction: $p(t) = \mathrm{sinc}(t/T)$

Proof of Sampling Theorem 1(2)



$$Z(t) = \sum_n X(nT)\, p(t - nT)$$

No need for Ψ.

Distorsion: $\varepsilon^2 = \mathrm{E}\big\{ (Z(t) - X(t))^2 \big\} = \mathrm{E}\{Z^2(t)\} - 2\mathrm{E}\{Z(t)X(t)\} + \mathrm{E}\{X^2(t)\}$

$$\mathrm{E}\{Z^2(t)\} = \mathrm{E}\left\{ \Big(\sum_n X(nT)\, p(t - nT) \Big)^2 \right\} = \sum_m \sum_n \mathrm{E}\{X(nT) X(mT)\}\, p(t - nT)\, p(t - mT)$$

$$= \sum_m \Big(\sum_n r_X(nT - mT)\, p(t - nT) \Big) p(t - mT)$$

$$\mathrm{E}\{Z(t)X(t)\} = \mathrm{E}\Big\{ \sum_n X(nT)\, p(t - nT)\, X(t) \Big\} = \sum_n \mathrm{E}\{X(nT) X(t)\}\, p(t - nT)$$

$$= \sum_n r_X(nT - t)\, p(t - nT)$$

$$\mathrm{E}\{X^2(t)\} = r_X(0)$$

Proof of Sampling Theorem 2(2)

We had:

$$\mathrm{E}\{Z^2(t)\} = \sum_m \left(\sum_n r_X(nT - mT)p(t-nT) \right) p(t-mT)$$

$$\mathrm{E}\{Z(t)X(t)\} = \sum_n r_X(nT - t)p(t-nT)$$

Try ideal reconstruction:

$$p(t) = \mathrm{sinc}\,(t/T)$$

From the deterministic case: $r_X(\tau) = \sum_m r_X(mT)p(\tau - mT)$

$$r_X(\tau - a) = \sum_n r_X(nT)p(\tau - a - nT) = \sum_n r_X(nT - a)p(\tau - nT) \quad \textbf{(1)}$$

With $\tau = t$ & $a = mT$ in (1), we get : $r_X(t - mT) = \sum_n r_X(nT - mT)p(t - nT)$ **(2)**

With $\tau = a = t$ in (1), we get : $r_X(0) = \sum_n r_X(nT - t)p(t - nT)$ **(3)**

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(2) & (3) $\Rightarrow$ $\mathrm{E}\{Z^2(t)\} = \sum_m r_X(t - mT)p(t - mT) = \sum_m r_X(mT - t)p(t - mT) = r_X(0)$

(3) $\Rightarrow$ $\mathrm{E}\{Z(t)X(t)\} = r_X(0)$

Result: $\varepsilon^2 = 0$

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