TSDT14 Signal Theory

Lecture 4

Estimation

Mikael Olofsson
Department of EE (ISY)
Div. of Communication Systems

LINKÖPINGS UNIVERSITET

DFT – Signal Analysis

Time-discrete signal with limited duration:

$$x[n] = 0 \quad \text{for} \quad n \notin \{0, 1, \ldots, N-1\}$$

Fourier transform: $X[\theta] = \sum_n x[n] e^{-j2\pi\theta n} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\theta n}$ cont. w. period 1.

DFT of length L: $X_L[k] = X[k/L] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{k}{L}n}$ for $k \in \{0, 1, \ldots, L-1\}$

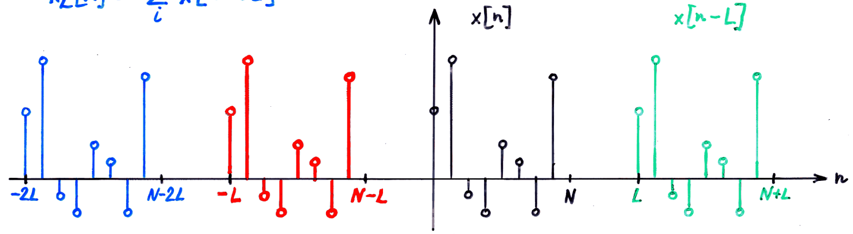
IDFT (inverse): $x_L[n] = \frac{1}{L} \cdot \sum_{k=0}^{L-1} X_L[k] e^{j2\pi\frac{k}{L}n} \Rightarrow x_L[n+L] = x_L[n]$ since $e^{j2\pi Ln/L} = e^{j2\pi \cdot n} = 1$

Relation to $x[n]$: $x_L[n] = \frac{1}{L} \sum_{k=0}^{L-1} \sum_{m=0}^{N-1} x[m] e^{-j2\pi\frac{k}{L}m} \cdot e^{j2\pi\frac{k}{L}n}$

$$= \sum_{m=0}^{N-1} x[m] \cdot \underbrace{\frac{1}{L} \sum_{k=0}^{L-1} e^{-j2\pi(m-n)\frac{k}{L}}}_{= \begin{cases} L, & m-n = 0 \bmod L \\ 0, & \text{elsewhere} \end{cases}} = \sum_{i=-\infty}^{\infty} x[n - i \cdot L]$$

DFT – Avoiding Aliasing

$$x_L[n] = \sum_i x[n - iL]$$



If $L < N$, then we get overlap and aliasing in the time domain.

Therefore: Demand $L \geq N$.

Note: $X_L[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{k}{L}n} = \sum_{n=0}^{L-1} x_L[n] e^{-j2\pi\frac{k}{L}n}$

DFT – Periodic Convolution

We are used to: $y[n] = (x * h)[n] \Leftrightarrow Y[\theta] = X[\theta] \cdot H[\theta]$

But we have: $y[n] = x[n] \cdot h[n] \Leftrightarrow Y[\theta] = \int_0^1 X[\phi] H[\theta - \phi] \, d\phi$

With DFT: $Y_L[k] = X_L[k] \cdot H_L[k] \Leftrightarrow$

$$y_L[n] = IDFT\{X_L[k] H_L[k]\} = \frac{1}{L} \sum_{k=0}^{L-1} X_L[k] H_L[k] e^{j2\pi\frac{k}{L} \cdot n}$$

$$= \frac{1}{L} \sum_{k=0}^{L-1} X_L[k] \cdot \sum_{m=0}^{L-1} h_L[m] \cdot e^{-j2\pi\frac{k}{L}m} \cdot e^{j2\pi\frac{k}{L}n}$$

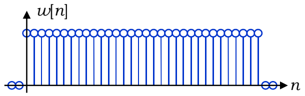
$$= \sum_{m=0}^{L-1} h_L[m] \cdot \underbrace{\frac{1}{L} \sum_{k=0}^{L-1} X_L[k] e^{j2\pi\frac{k}{L}(n-m)}}_{x_L[n-m]}$$

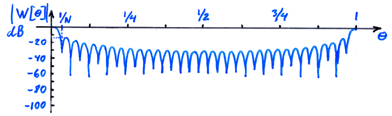
$$= \sum_{m=0}^{L-1} h_L[m] \, x_L[n-m]$$

And also: $y_L[n] = x_L[n] \cdot h_L[n] \Leftrightarrow Y_L[k] = \frac{1}{L} \sum_{m=0}^{L-1} X_L[m] H_L[k-m]$

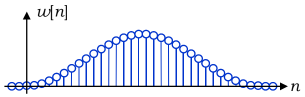
Examples of Windows, N=32

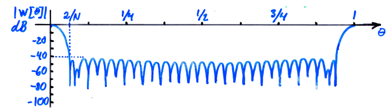
Rectangular window: $w[n] = 1, \quad n \in \{0,1,\ldots,N-1\}$



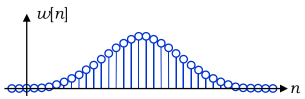


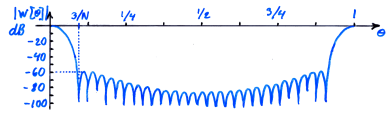
Hamming window: $w[n] = 0.54 - 0.46\cos(\frac{2\pi N}{N-1}), \quad n \in \{0,1,\ldots,N-1\}$





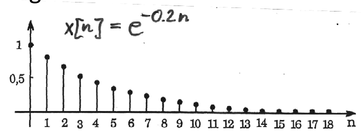
Blackman window: $w[n] = 0.42 - 0.5\cos(\frac{2\pi N}{N-1}) + 0.08\cos(\frac{4\pi N}{N-1}),$
$n \in \{0,1,\ldots,N-1\}$

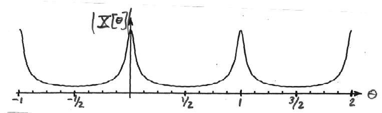




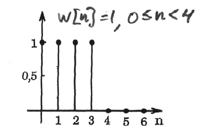
Using Windows

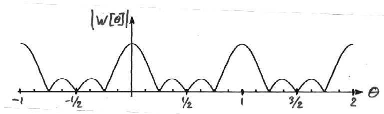
Signal:



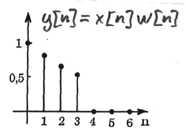


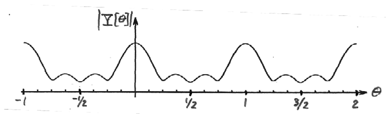
Rectangular window:





Result:





Ergodicity again

A WSS process: $m_X = E\{X(t)\}$

Time average of one realization: $m_T = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$

Time average of the process: $M_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$

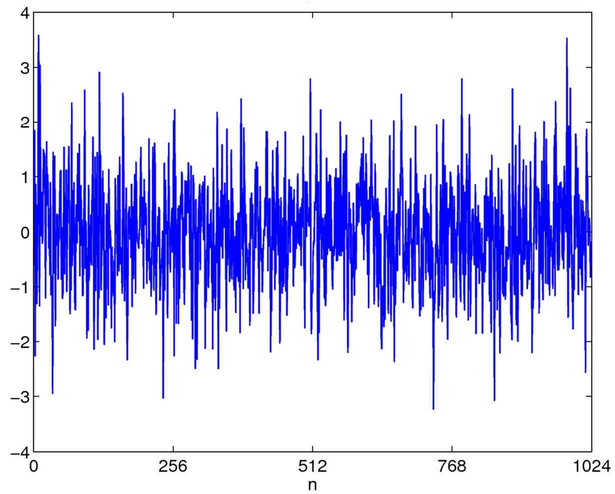
$$E\{M_T\} = E\left\{\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right\} = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = \frac{1}{2T}\int_{-T}^{T} m_X\,dt = m_X$$

Definition: If $\lim_{T\to\infty} E\{(M_T - m_X)^2\} = 0$ then $X(t)$ is said to be ergodic with respect to the mean, and we write

$$m_X = \underset{T\to\infty}{\text{l.i.m.}}\ \frac{1}{2T}\int_{-T}^{T} X(t)\,dt \qquad \text{(limes in mean)}$$

Interpretation: The time average of a realization is very close to the ensemble mean with a probability that is very close to 1 ($\to 1$, $T\to\infty$).

A Signal



Estimating a General Parameter

$X[n]$: An ergodic time-discrete process, with realization $x[n]$.

Estimation of a general parameter a_X, i.e. some ensemble average:

$$\hat{a}_X = g(x[0], x[1], \ldots, x[N-1])$$

Corresponding stochastic variable: $\hat{A}_X = g(X[0], X[1], \ldots, X[N-1])$

Bias (difference from actual value): $B = \mathrm{E}\{\hat{A}_X\} - a_X$

Unbiased estimate: $B = 0$

Asymptotically unbiased estimate: $B \to 0$ when $N \to \infty$

Variance: $\sigma_{\hat{A}_X} = \mathrm{E}\left\{\hat{A}_X{}^2\right\} - \mathrm{E}^2\{\hat{A}_X\}$

Quadratic error: $\varepsilon^2 = \sigma_{\hat{A}_X}{}^2 + B^2$

Consistent estimate: $\varepsilon^2 \to 0$ when $N \to \infty$

Blackman-Tukey's Estimate of the ACF 1(2)

ACF of a WSS process: $r_X[k] = \mathrm{E}\{X[n+k]X[n]\}$

Blackman-Tukey's method: $\hat{r}_X[k] = \dfrac{1}{N-|k|} \displaystyle\sum_{n=0}^{N-|k|-1} x[n+|k|]x[n]$

Mean: $\mathrm{E}\{\hat{r}_X[k]\} = r_X[k]$

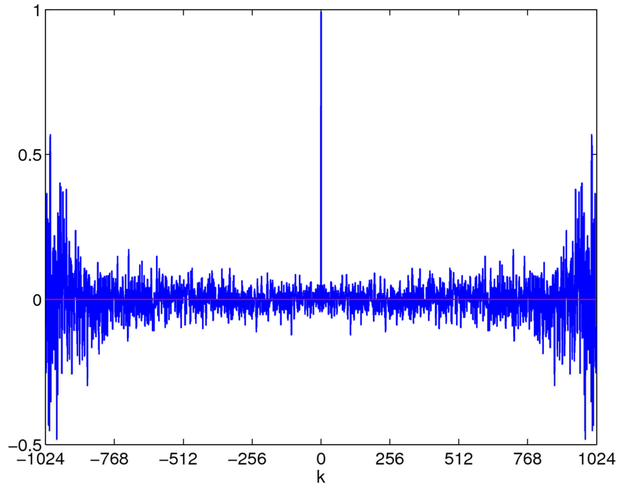
Variance, assuming a Gaussian process with mean zero:

$$|k| < N:\ \sigma_{\hat{r}_X}{}^2[k] = \frac{1}{N-|k|} \sum_{|m|<N-|k|} \left(1 - \frac{1}{N-|k|}\right)\left(r_X{}^2[m] + r_X[m+k]r_X[m-k]\right)$$

Fixed k: $\sigma_{\hat{r}_X}{}^2[k] \to 0$ when $N \to \infty$

Problem: Large k, Ex. $k = N-1$ $\quad \sigma_{\hat{r}_X}{}^2[k] \to r_X{}^2[0]$ when $N \to \infty$

Blackman-Tukey's Estimate of the ACF 2(2)



Bartlett's Estimate of the ACF 1(2)

Recall Blackman-Tukey: $\hat{r}_X[k] = \dfrac{1}{N-|k|} \displaystyle\sum_{n=0}^{N-|k|-1} x[n+|k|]x[n]$

Bartlett's method: $\hat{r}_X[k] = \dfrac{1}{N} \displaystyle\sum_{n=0}^{N-|k|-1} x[n+|k|]x[n]$

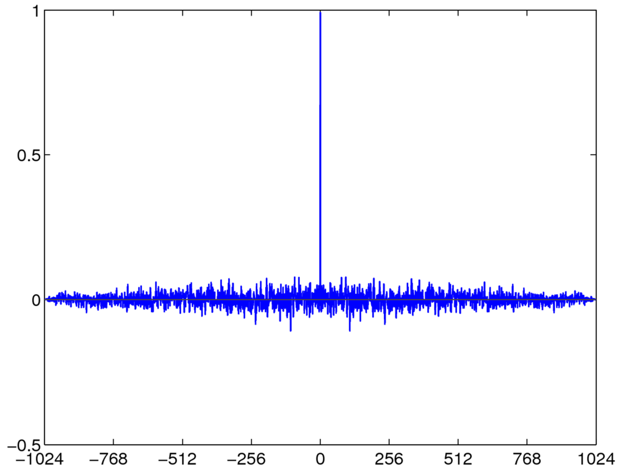
Mean: $\mathrm{E}\{\hat{r}_X[k]\} = \left(1 - \frac{|k|}{N}\right) r_X[k] \to r_X[k]$ when $N \to \infty$

Variance, assuming a Gaussian process with mean zero:

$$|k| < N:\ \sigma_{\hat{r}_X}{}^2[k] = \frac{1}{N} \sum_{|m|<N-|k|} \left(1 - \frac{|k|+|m|}{N}\right)\left(r_X{}^2[m] + r_X[m+k]r_X[m-k]\right)$$

All k: $\sigma_{\hat{r}_X}{}^2[k] \to 0$ when $N \to \infty$

Bartlett's Estimate of the ACF 2(2)



Estimating the PSD 1(2)

PSD of a WSS process: $R_X[\theta] = \mathcal{F}\{r_X[k]\}$

Observed sequence: $x[n]$

Considered part of sequence: $x_N[n] = \begin{cases} x[n], & 0 \le n < N \\ 0 & \text{elsewhere} \end{cases}$

Bartlett's estimate of the ACF: $\hat{r}_X[k] = \frac{1}{N}\sum_{m=-\infty}^{\infty} x_N[m+k]x_N[m]$

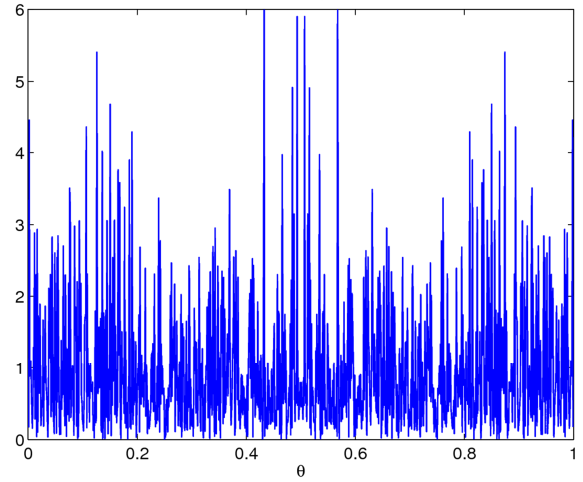
Estimate of the PSD: $\hat{R}_X[\theta] = \mathcal{F}\{\hat{r}_X[k]\}$

$$= \sum_{k=-\infty}^{\infty}\frac{1}{N}\sum_{m=-\infty}^{\infty} x_N[m+k]x_N[m]e^{-j2\pi\theta k} = \frac{1}{N}\sum_{m=-\infty}^{\infty} x_N[m]\sum_{k=-\infty}^{\infty} x_N[m+k]e^{-j2\pi\theta k}$$

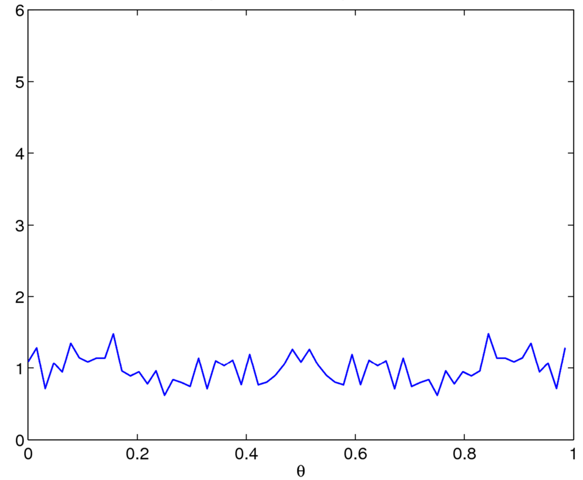
$$= \frac{1}{N}X_N[\theta]\sum_{m=-\infty}^{\infty} x_N[m]e^{j2\pi\theta m} = \frac{1}{N}X_N[\theta]X_N{}^*[\theta] = \frac{1}{N}|X_N[\theta]|^2$$

Called periodogram

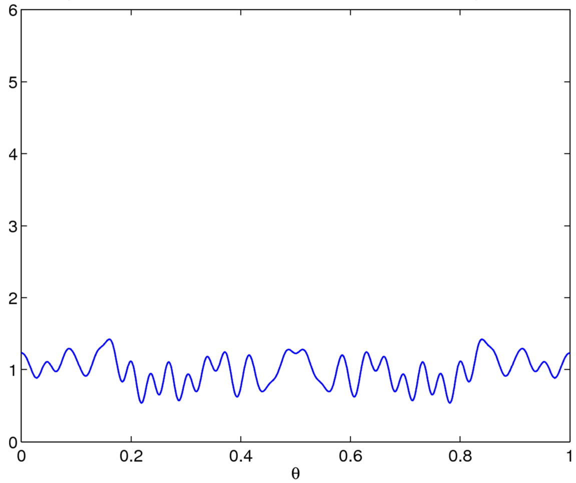
Periodogram



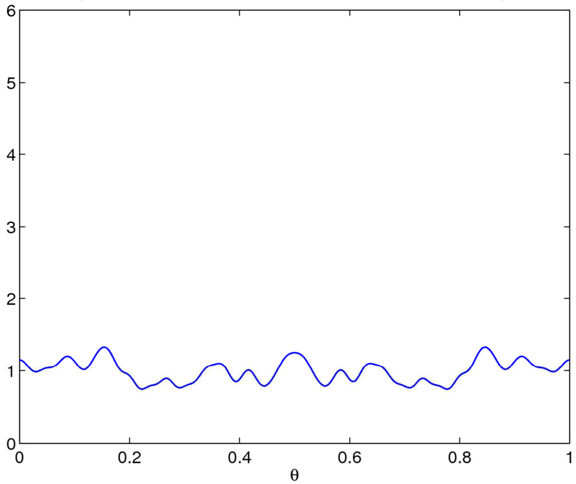
Averaged Periodogram



Estimated PSD using Rectangular Window

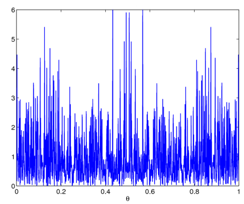


Estimated PSD using Hamming Window

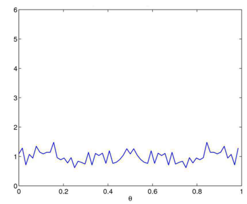


Smoothing – Overview

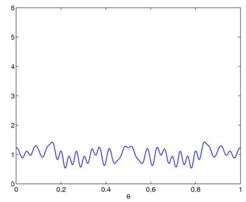
Raw periodogram



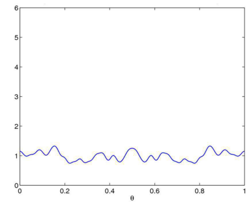
Averaged periodograms



Smoothing - Rectangular



Smoothing - Hamming



Mikael Olofsson
ISY/CommSys

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