TSDT14 Signal Theory

Lecture 6

Saturation, Quantization and Poisson Processes

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Non-Linearities

From Tables & Formulas, Page 15.

Situation

The input to a momentary non-linearity is a Gaussian process $X(t)$ with mean $m_X = 0$ and auto-correlation function $r_X(\tau)$.

Identities

$Y(t)$	$r_Y(\tau)$
$X^2(t)$	$2r_X^2(\tau) + r_X^2(0)$
$X^3(t)$	$6r_X^3(\tau) + 9r_X^2(0)r_X(\tau)$
$X^4(t)$	$24r_X^4(\tau) + 72r_X^2(0)r_X^2(\tau) + 9r_X^4(0)$
$X^5(t)$	$120r_X^5(\tau) + 600r_X^2(0)r_X^3(\tau) + 225r_X^4(0)r_X(\tau)$
$\begin{cases} X, & X \geq 0, \\ 0, & X < 0. \end{cases}$	$\frac{r_X(\tau)}{4} + \frac{1}{2\pi}\left[\sqrt{r_X^2(0) - r_X^2(\tau)} + r_X(\tau)\arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right)\right]$ $= \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^2(\tau)}{4\pi r_X(0)} + \ldots$
$\mathrm{sgn}(X)$	$\frac{2}{\pi}\arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right)$

These relations are valid for the time-discrete case as well.

Half-Wave Rectifier

From the table:

$Y(t)$	$r_Y(\tau)$
$\begin{cases} X, & X \geq 0, \\ 0, & X < 0. \end{cases}$	$\frac{r_X(\tau)}{4} + \frac{1}{2\pi}\left[\sqrt{r_X^2(0) - r_X^2(\tau)} + r_X(\tau)\arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right)\right]$ $= \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^2(\tau)}{4\pi r_X(0)} + \ldots$

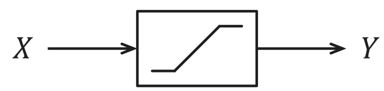
Complete Maclaurin expansion:

$$r_Y(\tau) = \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^2(\tau)}{4\pi r_X(0)} + \sum_{n=2}^{\infty} \frac{(2n-3)!! \cdot r_X(0)}{2\pi \cdot (2n-1) \cdot (2n)!!} \left(\frac{r_X(\tau)}{r_X(0)}\right)^{2n}$$

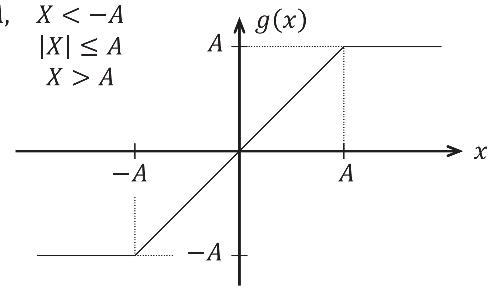
$n!!$ is semi-factorial (product of every second positive integer):

$$6!! = 2 \cdot 4 \cdot 6 = 48 \qquad \text{and} \qquad 7!! = 1 \cdot 3 \cdot 5 \cdot 7 = 105.$$

Saturation

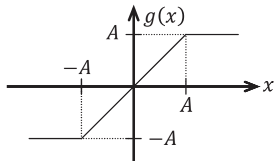


$$Y = g(X) = \begin{cases} -A, & X < -A \\ X, & |X| \leq A \\ A & X > A \end{cases}$$



Saturation PDF

$$Y = g(X) = \begin{cases} -A, & X < -A \\ X, & |X| \le A \\ A & X > A \end{cases}$$



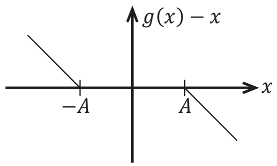
$$p_1 = \Pr\{X < -A\} = \int_{-\infty}^{-A} f_X(x)\,dx \qquad p_2 = \Pr\{X > A\} = \int_{A}^{\infty} f_X(x)\,dx$$

$$\tilde{f}_Y(y) = f_X(y) \cdot \mathrm{rect}\left(\frac{y}{2A}\right) = \begin{cases} f_X(y), & |y| \le A \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = p_1\delta(y + A) + \tilde{f}_Y(y) + p_2\delta(y - A)$$

Saturation Error

$$S = Y - X = g(X) - X = \begin{cases} -A - X, & X < -A \\ 0, & |X| \le A \\ A - X & X > A \end{cases}$$

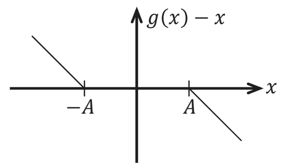


$$p_3 = \Pr\{|X| \le A\} = \int_{-A}^{A} f_X(x)\,dx = 1 - p_1 - p_2$$

$$\tilde{f}_S(s) = \begin{cases} f_X(-A - s) & s > 0 \\ 0 & s = 0 \\ f_X(A - s) & s < 0 \end{cases}$$

$$f_S(s) = \tilde{f}_S(s) + p_3\delta(s)$$

Saturation Distorsion



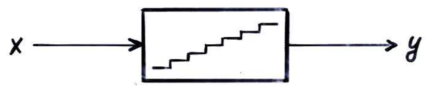
$$P_S = \mathrm{E}\{S^2\} = \int_{-\infty}^{\infty} s^2 f_S(s)\,ds = \int_{-\infty}^{0} s^2 f_X(A - s)\,ds + \int_{0}^{\infty} s^2 f_X(-A - s)\,ds$$

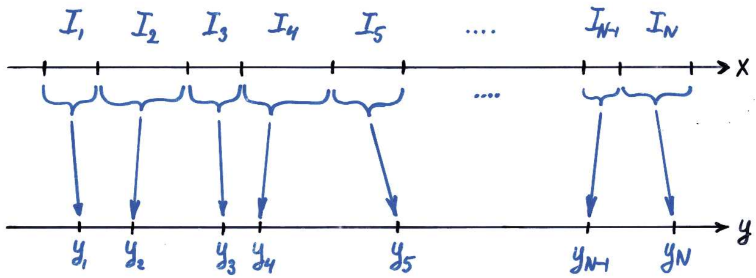
$$= \int_{A}^{\infty} (x - A)^2 f_X(x)\,dx + \int_{-\infty}^{-A} (x + A)^2 f_X(x)\,dx$$

SDR – Signal-to-Distorsion Ratio

$$\mathrm{SDR} = \frac{P_X}{P_S} \qquad \mathrm{SDR}_{\mathrm{dB}} = 10 \cdot \log_{10}\left(\frac{P_X}{P_S}\right)$$

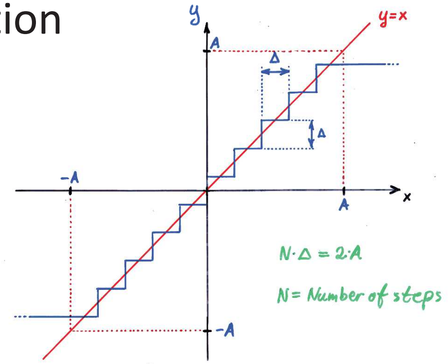
Quantization Principles





Uniform Quantization

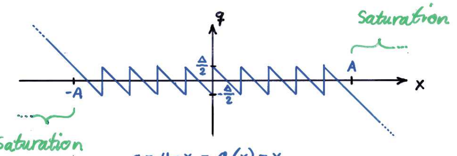
$$y = g(x) = \begin{cases} A - \frac{\Delta}{2}, & x > A \\ \frac{\Delta}{2} + \left\lfloor \frac{x}{\Delta} \right\rfloor \cdot \Delta, & |x| \le A \\ -A + \frac{\Delta}{2}, & x < -A \end{cases}$$



$N \cdot \Delta = 2 \cdot A$

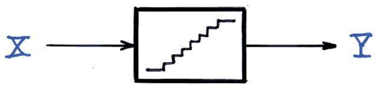
N = Number of steps

Quantization error:



$q = y - x = g(x) - x.$

Quantization Distorsion 1(2)



The error: $Q = Y - X = g(X) - X$

Quantization distorsion:

$$P_Q = E\{Q^2\} = E\{(g(X) - X)^2\} = \int_{-\infty}^{\infty} (g(x) - x)^2 f_X(x)\, dx$$

Assumptions:

1. No saturation: $f_X(x) = 0$ for $|x| \ge A$
2. Nice distribution: $f_X(x)$ continuous for $|x| < A$
3. Small Δ: $f_X(x)$ approx. const. in intervals of length Δ.

Quantization Distorsion 2(2)

$$P_Q = \int_{-A}^{A} (g(x) - x)^2 f_X(x)\, dx = \sum_{k=1}^{N} \int_{y_k - \frac{\Delta}{2}}^{y_k + \frac{\Delta}{2}} (y_k - x)^2 \cdot f_X(x)\, dx = \Big/ \begin{array}{l} u = x - y_k \\ du = dx \end{array} \Big/$$

$$= \sum_{k=1}^{N} \int_{-\Delta/2}^{\Delta/2} u^2 f_X(y_k + u)\, du \underset{\substack{\Delta \text{ small} \\ u \text{ small}}}{\approx} \sum_{k=1}^{N} \int_{-\Delta/2}^{\Delta/2} u^2 \cdot f_X(y_k)\, du$$

$$= \sum_{k=1}^{N} f_X(y_k) \underbrace{\int_{-\Delta/2}^{\Delta/2} u^2 du}_{=\Delta^3/12} = \frac{\Delta^2}{12} \sum_{k=1}^{N} \Delta \cdot f_X(y_k) \approx \frac{\Delta^2}{12} \underbrace{\sum_{k=1}^{N} Pr\{X \in I_k\}}_{=1} = \frac{\Delta^2}{12}$$

Error distribution: Approx. uniformly distr. over $[-\frac{\Delta}{2}, \frac{\Delta}{2})$

Generally without saturation:

$P_Q \le \frac{\Delta^2}{4}$ since $|Q| \le \frac{\Delta}{2}$.

SDR for Uniform Quantization

Still limited to [-*A*,*A*] and nice enough distribution.

$$P_Q = \mathrm{E}\left\{Q^2\right\} = \int_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta}\, dq = \frac{\Delta^2}{12} = \frac{A^2}{3N^2} \qquad \Rightarrow \qquad \mathrm{SDR} = \frac{P_X}{P_Q} = \frac{3P_X}{A^2} N^2 = \frac{3P_X}{A^2} 2^{2n}$$

$$\mathrm{SDR_{dB}} = 10 \log_{10}(\mathrm{SDR}) = 10 \log_{10}\left(\frac{3P_X}{A^2}\right) + n \cdot 20 \log_{10}(2) \approx 10 \log_{10}\left(\frac{3P_X}{A^2}\right) + 6n.$$

Example: Uniform distribution over [-*A*,*A*].

$$P_X = \mathrm{E}\left\{X^2\right\} = \int_{-A}^{A} x^2 \frac{1}{2A}\, dx = \frac{A^2}{3},$$

$$\mathrm{SDR_{dB}} \approx 10 \log_{10}\left(\frac{3A^2/3}{A^2}\right) + 6n = 10 \log_{10}(1) + 6n = 6n$$

SDR for Uniform Quantization with Saturation

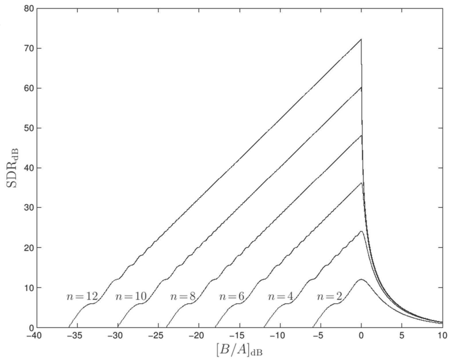
Uniform distribution over [-*B*,*B*].

$$P_X = \mathrm{E}\left\{X^2\right\} = \int_{-B}^{B} x^2 \frac{1}{2B}\, dx = \frac{B^2}{3}.$$

Q and S uncorrelated:

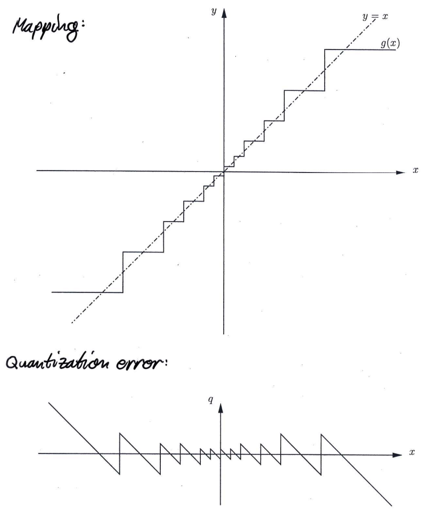
$$P_{Q+S} = P_Q + P_S.$$

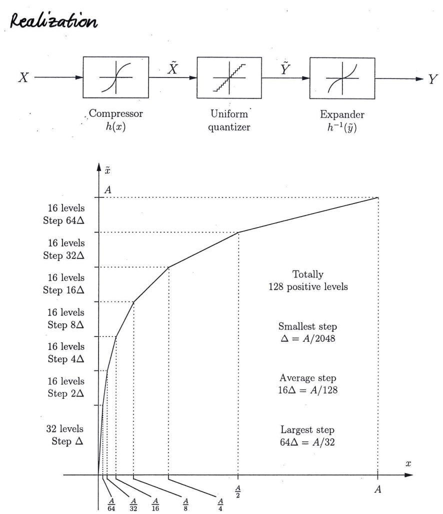
$$\mathrm{SDR} = \frac{P_X}{P_Q + P_S},$$



$$P_{Q+S} = \begin{cases} \frac{(2k+1)(\Delta/2)^3 + (B-(2k+1)\Delta/2)^3}{3B}, & k\Delta \le B < (k+1)\Delta,\ k \in \{0,1,...,N/2-2\} \\ \frac{A-\Delta/2}{B}\cdot\frac{\Delta^2}{12} + \frac{(B-A+\Delta/2)^3}{3B} & B \ge A - \frac{\Delta}{2} \end{cases}$$

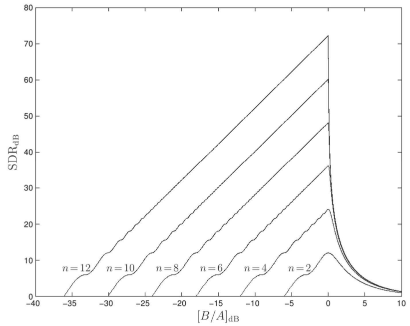
Non-Uniform Quantization



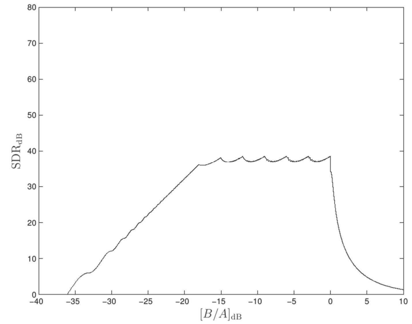


SDR for Non-Uniform Quantization

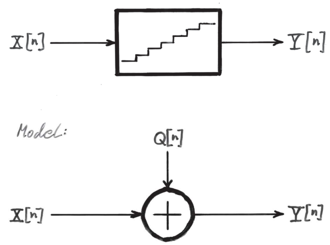
Uniform Quantization



Non-Uniform Quantization



Modelling Quantization of a Stochastic Process



Model:

- The quantization noise is white
- The input and the quantization noise are uncorrelated.

Reason:

- It is almost true under reasonable assumptions.
- It gets more true with smaller quantization step.

The Quantization Noise is Almost White 1(4)

Quantization Noise: $Q[n] = Y[n] - X[n]$.

Model of PSD: $R_Q[\theta] = \dfrac{\Delta^2}{12}$

Assumptions:

1. that $f_{X[0],X[k]}(x_0, x_k) = 0$ holds for $|x_0| \geq A$ and for $|x_k| \geq A$,
2. that $f_{X[0],X[k]}(x_0, x_k)$ is three times differentiable inside the square $|x_0| < A$, $|x_k| < A$,
3. that $f_{X[0],X[k]}(-x_0, -x_k) = f_{X[0],X[k]}(x_0, x_k)$ holds.

Objective:

Show $\dfrac{r_Q[k]}{r_Q[0]} \to \delta[k], \quad \text{when } \Delta \to 0,$

The Quantization Noise is Almost White 2(4)

ACF of the Quantization Noise for $k \neq 0$:

$$r_Q[k] = \iint_{-\Delta/2}^{\Delta/2} q_0 q_k f_{Q[0],Q[k]}(q_0, q_k)\, dq_0\, dq_k$$

2-D PDF of the Quantization Noise:

$$f_{Q[0],Q[k]}(q_0, q_k) = $$

$$= \begin{cases} \sum_{i_0=1}^{\frac{2A}{\Delta}} \sum_{i_k=1}^{\frac{2A}{\Delta}} f_{X[0],X[k]}\left(q_0 - A - \frac{\Delta}{2} + i_0\Delta, q_k - A - \frac{\Delta}{2} + i_k\Delta\right), & |q_0| < \frac{\Delta}{2}, |q_k| < \frac{\Delta}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$

2-D Taylor series expansion of $f_{X[0],X[k]}\left(q_0 - A - \frac{\Delta}{2} + i_0\Delta, q_k - A - \frac{\Delta}{2} + i_k\Delta\right)$ gives us:

$$f_{Q[0],Q[k]}(q_0, q_k) = \frac{K_1 + K_2 q_0 + K_3 q_k + K_4 q_0^2 + K_5 q_0 q_k + K_6 q_k^2 + K_7 (q_0^2 + q_k^2)^{3/2}}{\Delta^2}$$

The Quantization Noise is Almost White 3(4)

What about all those coefficients? Symmetry:

$$f_{X[0],X[k]}(-x_0, -x_k) = f_{X[0],X[k]}(x_0, x_k) \quad \Rightarrow \quad K_2 = K_3 = 0$$

Result:

$$r_Q[k] = \iint_{-\Delta/2}^{\Delta/2} q_0 q_k \frac{K_1 + K_4 q_0^2 + K_5 q_0 q_k + K_6 q_k^2 + K_7 (q_0^2 + q_k^2)^{3/2}}{\Delta^2}\, dq_0\, dq_k.$$

Observation:

$$\iint_{-\Delta/2}^{\Delta/2} q_0 q_k\, dq_0\, dq_k = \iint_{-\Delta/2}^{\Delta/2} q_0^3 q_k\, dq_0\, dq_k = \iint_{-\Delta/2}^{\Delta/2} q_0 q_k^3\, dq_0\, dq_k = 0$$

Result:

$$r_Q[k] = \iint_{-\Delta/2}^{\Delta/2} \frac{K_5 q_0^2 q_k^2 + K_7 q_0 q_k (q_0^2 + q_k^2)^{3/2}}{\Delta^2}\, dq_0\, dq_k.$$

The Quantization Noise is Almost White 4(4)

We had:

$$r_Q[k] = \iint_{-\Delta/2}^{\Delta/2} \frac{K_5 q_0^2 q_k^2 + K_7 q_0 q_k (q_0^2 + q_k^2)^{3/2}}{\Delta^2}\, dq_0\, dq_k.$$

Upper bound:

$$\left|r_Q[k]\right| < K\Delta^4 \quad \Rightarrow \quad \frac{\left|r_Q[k]\right|}{r_Q[0]} < 12K\Delta^2.$$

Result:

$$\frac{r_Q[k]}{r_Q[0]} \to \delta[k], \quad \text{when } \Delta \to 0.$$

Conclusion:

Almost white. Closer to white as Δ decreases.

Input & Quantization Noise Almost Uncorrelated

Same assumptions:

1. that $f_{X[0],X[k]}(x_0, x_k) = 0$ holds for $|x_0| \geq A$ and for $|x_k| \geq A$,
2. that $f_{X[0],X[k]}(x_0, x_k)$ is three times differentiable inside the square $|x_0| < A$, $|x_k| < A$,
3. that $f_{X[0],X[k]}(-x_0, -x_k) = f_{X[0],X[k]}(x_0, x_k)$ holds.

Normalized cross-covariance:

$$\rho_{XQ}[k] = \frac{\mathrm{Cov}\{X[0], Q[k]\}}{\sigma_X \sigma_Q} = \frac{\mathrm{E}\{(X[0] - m_X)(Q[k] - m_Q)\}}{\sigma_X \sigma_Q}$$

Objective:

Show $\rho_{XQ}[k] \to 0, \quad \text{when } \Delta \to 0, \quad \text{for all } k.$

Modelling Quantization Noise

Observation:

$$f_{X[0],X[k]}(-x_0, -x_k) = f_{X[0],X[k]}(x_0, x_k)$$

$\Rightarrow \quad f_X(x)$ & $f_Q(q)$ even $\quad \Rightarrow \quad m_Q = m_X = 0$

$$\Rightarrow \quad \rho_{XQ}[k] = \frac{r_{XQ}[k]}{\sigma_X \sigma_Q} = \frac{\mathrm{E}\{X[0]Q[k]\}}{\sigma_X \sigma_Q}$$

Similar reasoning as before:

$$\left|\rho_{XQ}[k]\right| < \sqrt{12} K \Delta / \sigma_X \quad \Rightarrow \quad \rho_{XQ}[k] \to 0, \quad \text{when } \Delta \to 0, \quad \text{for all } k.$$

Conclusion:

Almost uncorrelated. Less correlated as Δ decreases.

Quantization – ACF & PSD Relations

Assumptions:

- $Q[n]$ is uniformly distributed on $[-\Delta/2, \Delta/2)$.
- $Q[n]$ is a white process.
- $Q[n]$ and $X[n]$ are uncorrelated.

ACF of output:

$$r_Y[k] = \mathrm{E}\{Y[n]Y[n+k]\} = \mathrm{E}\Big\{\big(X[n] + Q[n]\big)\big(X[n+k] + Q[n+k]\big)\Big\}$$
$$= \mathrm{E}\{X[n]X[n+k]\} + \mathrm{E}\{X[n]Q[n+k]\} + \mathrm{E}\{Q[n]X[n+k]\} + \mathrm{E}\{Q[n]Q[n+k]\}.$$

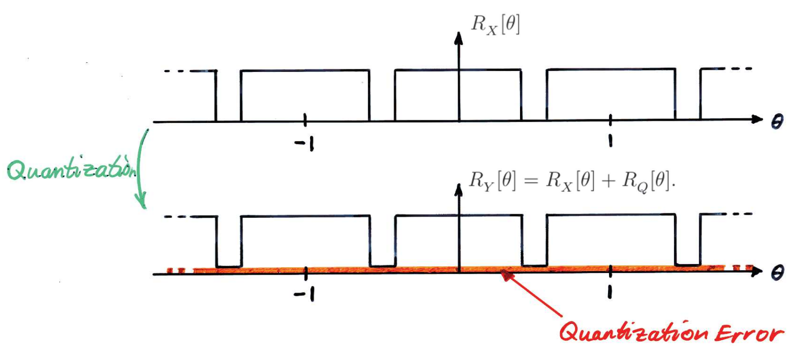
Uncorrelated processes:

$$\mathrm{E}\{X[n]Q[n+k]\} = \mathrm{E}\{Q[n]X[n+k]\} = m_X m_Q = 0,$$

Result:

$$r_Y[k] = r_X[k] + r_Q[k]. \qquad R_Y[\theta] = R_X[\theta] + R_Q[\theta].$$

Quantization – Power-Spectral Densities



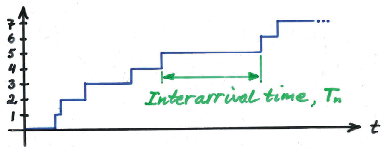
Poisson Processes 1(2)

A counting process, $X(t)$. Time continuous & amplitude discrete.
Counting the number of arrivals so far.

Examples:

- Customers entering a shop.
- Cars passing by
- Radio-active decay
- Light (counting photons)
- Packets in a network

A realization:



Properties of a stationary Poisson process:

$X(0) = 0$

$X(t_1) \leq X(t_2)$ if $t_1 \leq t_2$.

$Pr\{X(t) = k\} = \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t}, \quad t \geq 0, \; k \in N$

$Pr\{X(t+\tau) - X(t) = k\} = \frac{(\lambda \tau)^k}{k!} e^{-\lambda \tau}, \quad t \geq 0 \;\&\; \tau \geq 0, \; k \in N$

Interarrival times:

T_m & T_n indep. for $m \neq n$

$f_{T_n}(t_n) = \lambda \cdot e^{-\lambda t_n}, \quad t_n \geq 0$

$F_{T_n}(t_n) = 1 - e^{-\lambda t_n}, \quad t_n \geq 0$

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Poisson Processes 2(2)

Expectation: $E\{X(t)\} = \sum_{k=0}^{\infty} k Pr\{X(t) = k\} = \lambda t$ (λ intensity)

Variance: $Var\{X(t)\} = \lambda t$ — for $t \geq 0$

Power: $E\{X^2(t)\} = E^2\{X(t)\} + Var\{X(t)\} = \lambda t(1 + \lambda t)$

Increments: $X(t+\tau) - X(t)$. Nonoverlapping increments are independent.

ACF $(0 \leq t_1 \leq t_2)$: $r_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{X(t_1)(X(t_2) - X(t_1) + X(t_1))\}$

$= E\{(X(t_1) - X(0))(X(t_2) - X(t_1))\} + E\{X^2(t_1)\}$

Indep incr, $0 \leq t_1 \leq t_2$

$= E\{X(t_1) - X(0)\} \cdot E\{X(t_2) - X(t_1)\} + E\{X^2(t_1)\}$

$= \lambda t_1 (\lambda t_2 - \lambda t_1) + \lambda t_1 (1 + \lambda t_1) = \lambda t_1 (1 + \lambda t_2)$

ACF $(0 \leq t_2 \leq t_1)$: $r_X(t_1, t_2) = \lambda t_2 (1 + \lambda t_1)$ (similarity)

ACF (total): $r_X(t_1, t_2) = \lambda \cdot \min\{t_1, t_2\} + \lambda^2 t_1 \cdot t_2$