TSKS01 Digital Communication
Lecture 3

Signals as Vectors

Emil Björnson

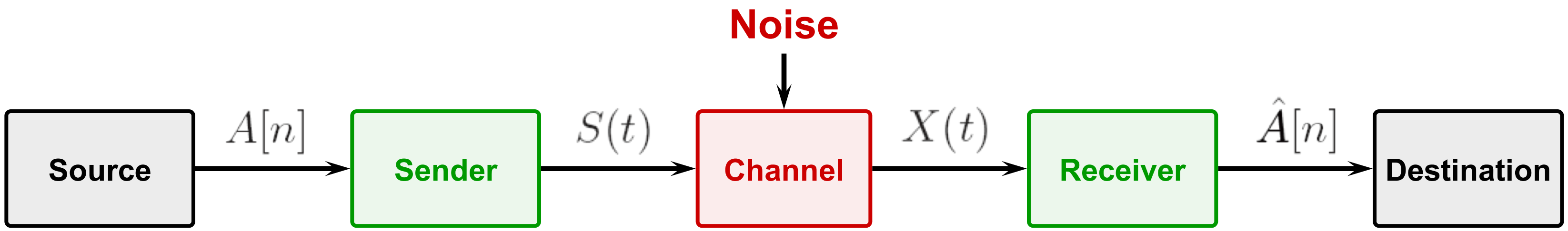
Department of Electrical Engineering (ISY)

Division of Communication Systems

COMMUNICATION SYSTEMS

LINKÖPING UNIVERSITY

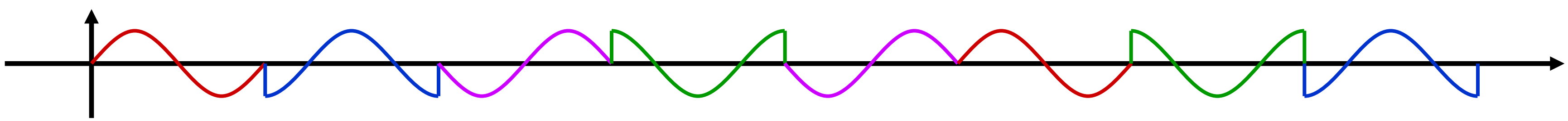
Situation: Additive Noise Channel



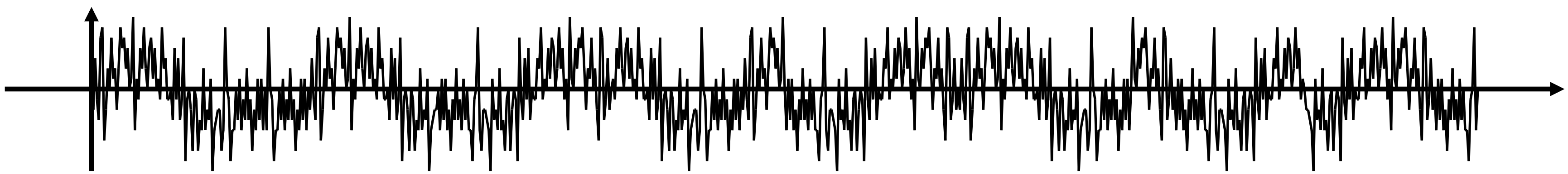
One message every T seconds.

00 10 11 01 11 00 01 10

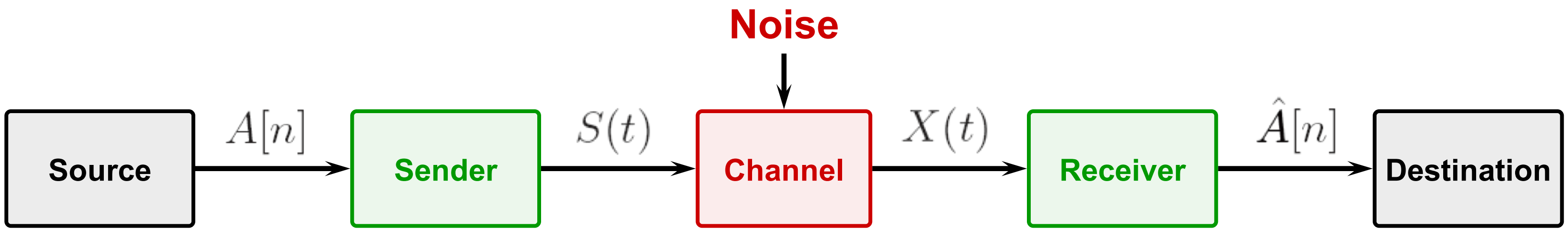
Represented using time-delimited signals. Duration T seconds ($T \sim 1/\mathrm{bandwidth}$)



The channel adds noise to the signal.



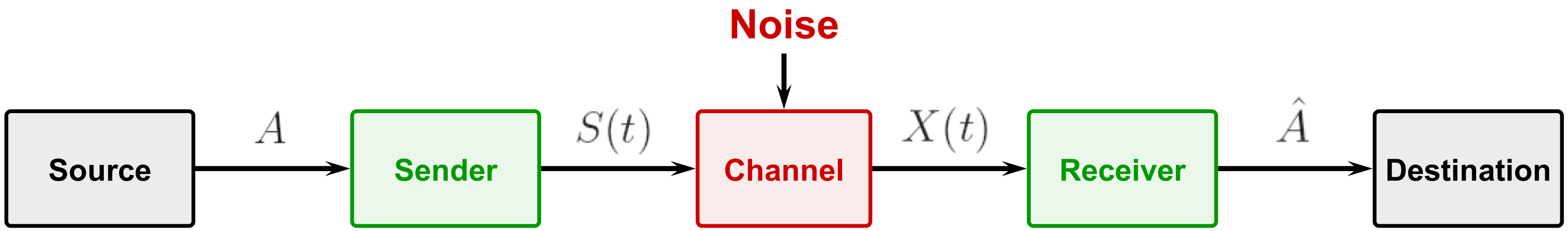
Analyze Only One Message at Time



Reasons:

- Different samples in $A[n]$ are assumed independent
  $\Rightarrow$ No information gained about one sample from another.
- Pulse function satisfies Nyquist criterion
  $\Rightarrow$ No intersymbol interference due to modulation.
- The channel does not filter the signal
  $\Rightarrow$ No intersymbol interference due to filtering.
- Different samples of WGN are independent
  $\Rightarrow$ Noise from non-overlapping intervals are independent.

Notation



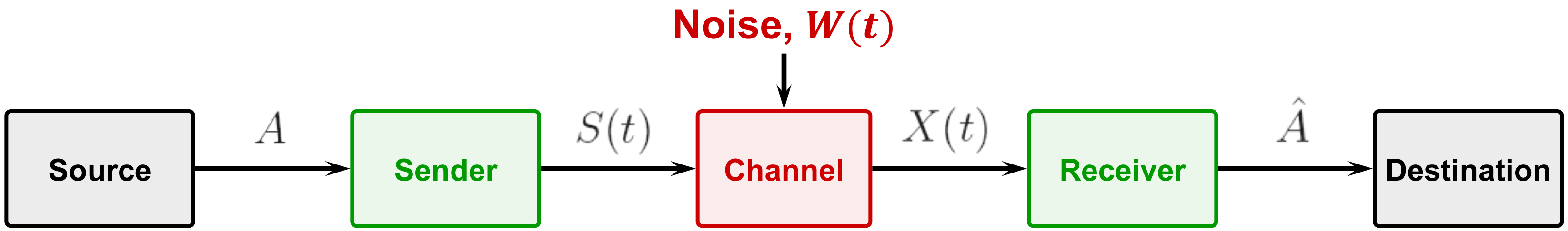
Message set: $\{a_i\}_{i=0}^{M-1}$ Size: M

Signal set: $\{s_i(t)\}_{i=0}^{M-1}$

Mapping: $a_i \rightarrow s_i(t)$ for $i \in \{0,1,\ldots,M-1\}$

Signal energy: $E_i = \int_{-\infty}^{\infty} s_i^2(t)dt < \infty\,,\qquad i \in \{0,1,\ldots,M-1\}$

The Channel and the Receiver



AWGN channel: Additive White Gaussian Noise

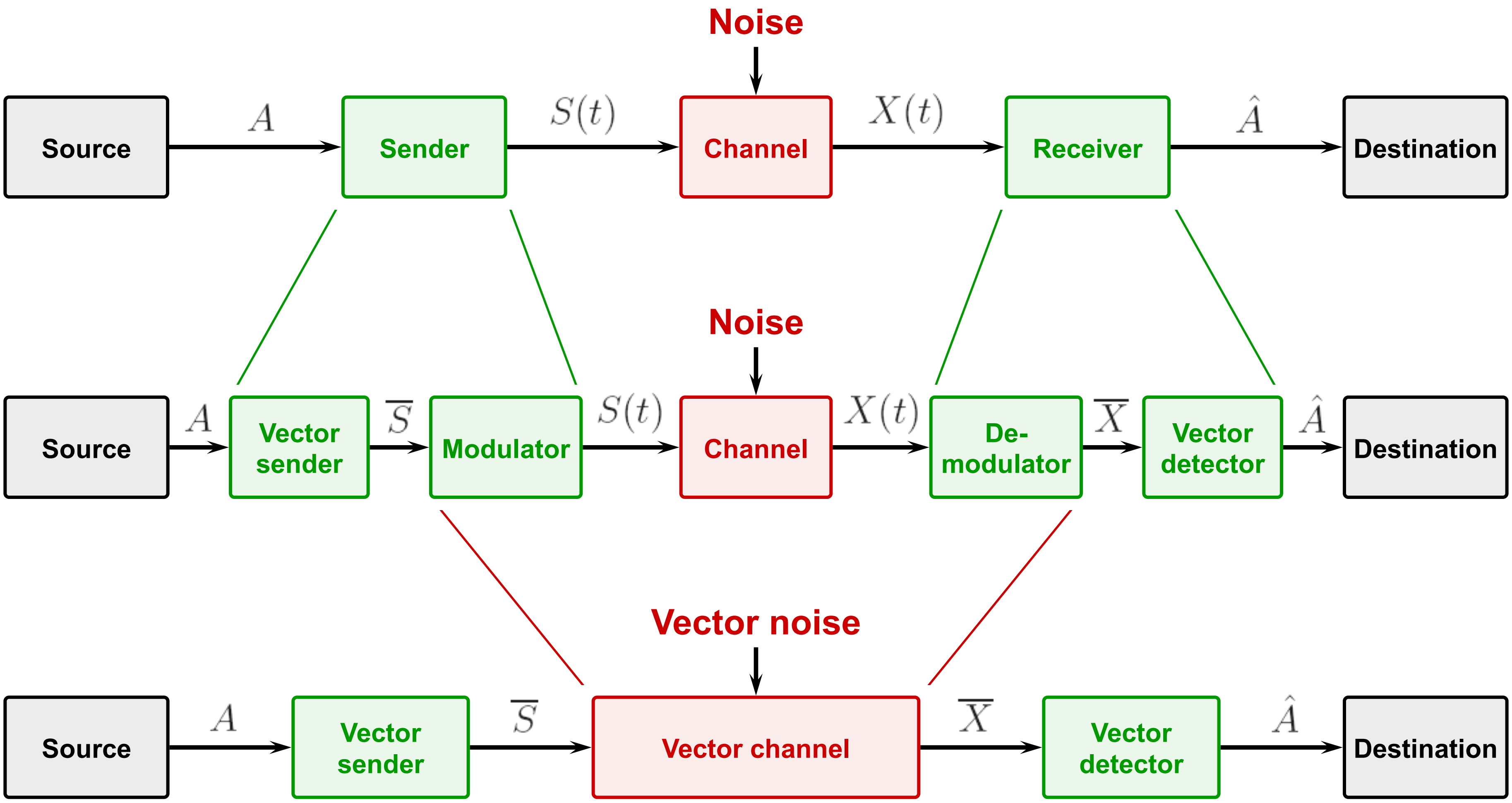
Channel output: $X(t) = s_i(t) + W(t)$

Receiver: Observes $X(t)$. Generates estimate $\hat{A}$.

Goal: Minimize $\Pr\{\hat{A} \neq A\}$.

This lecture concentrates on the sender

Where are we heading?



Function Spaces – Vector Spaces

M signals, $s_i(t),\ i \in \{0, 1, \dots, M-1\}$ span N dimensions, $N \leq M$.

Ortho-normal (ON) basis: $\{\phi_j(t)\}_{j=0}^{N-1}$ (spans the same dimensions)

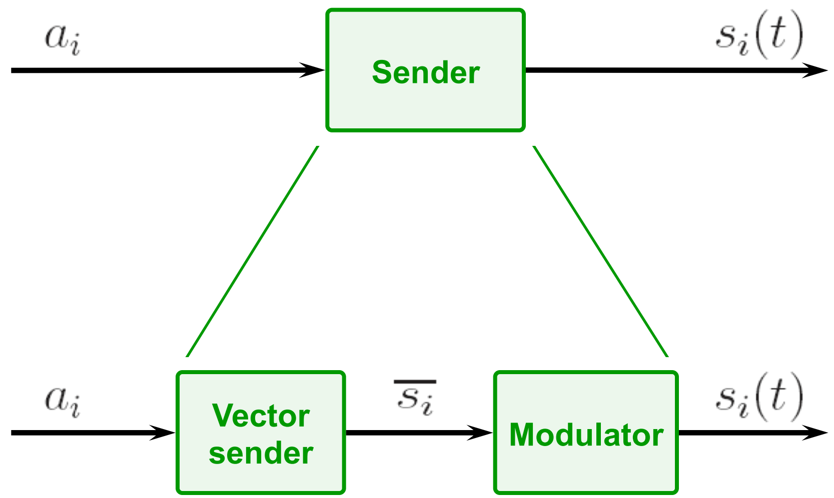
$$\int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)dt = \delta_{i,j} \triangleq \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

The signals expressed using this basis:

$$s_i(t) = \sum_{j=0}^{N-1} s_{i,j}\phi_j(t),\ i \in \{0, 1, \dots, M-1\}$$

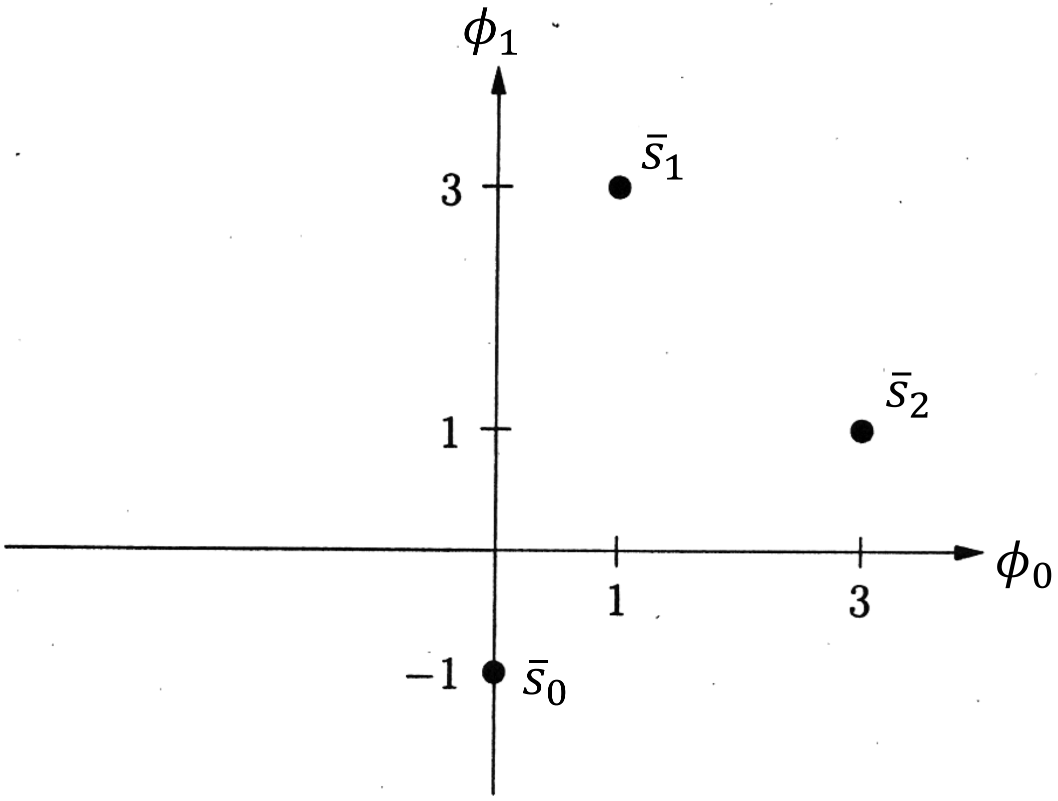
Corresponding vectors: $\overline{s_i} = \begin{pmatrix} s_{i,0} \\ s_{i,1} \\ \vdots \\ s_{i,N-1} \end{pmatrix}, \quad i \in \{0, 1, \dots, M-1\}$

Result – Split the Sender

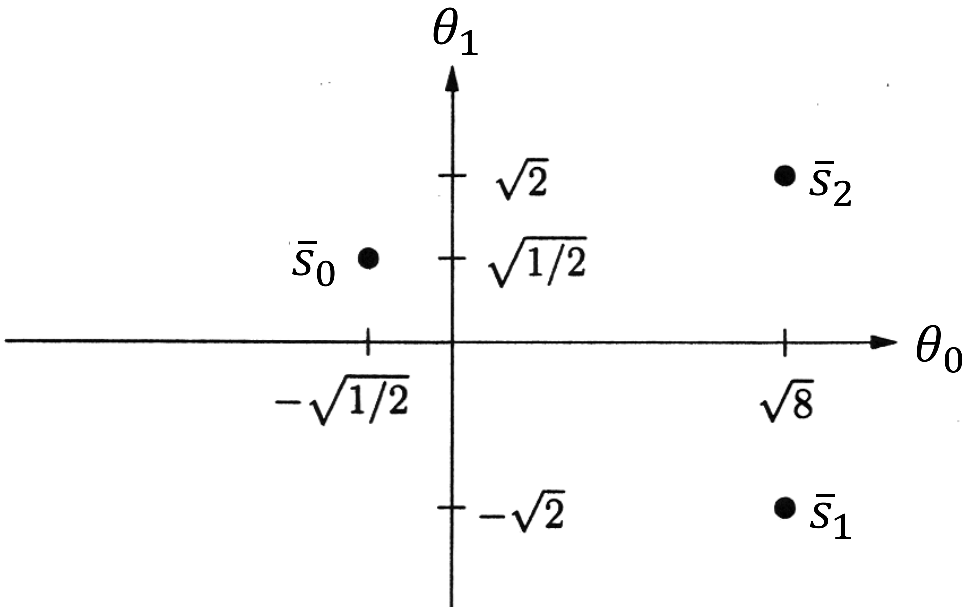


$$a_i \longrightarrow \begin{pmatrix} s_{i,0} \\ s_{i,1} \\ \vdots \\ s_{i,N-1} \end{pmatrix} \longrightarrow \sum_{j=0}^{N-1} s_{i,j}\,\phi_j(t)$$

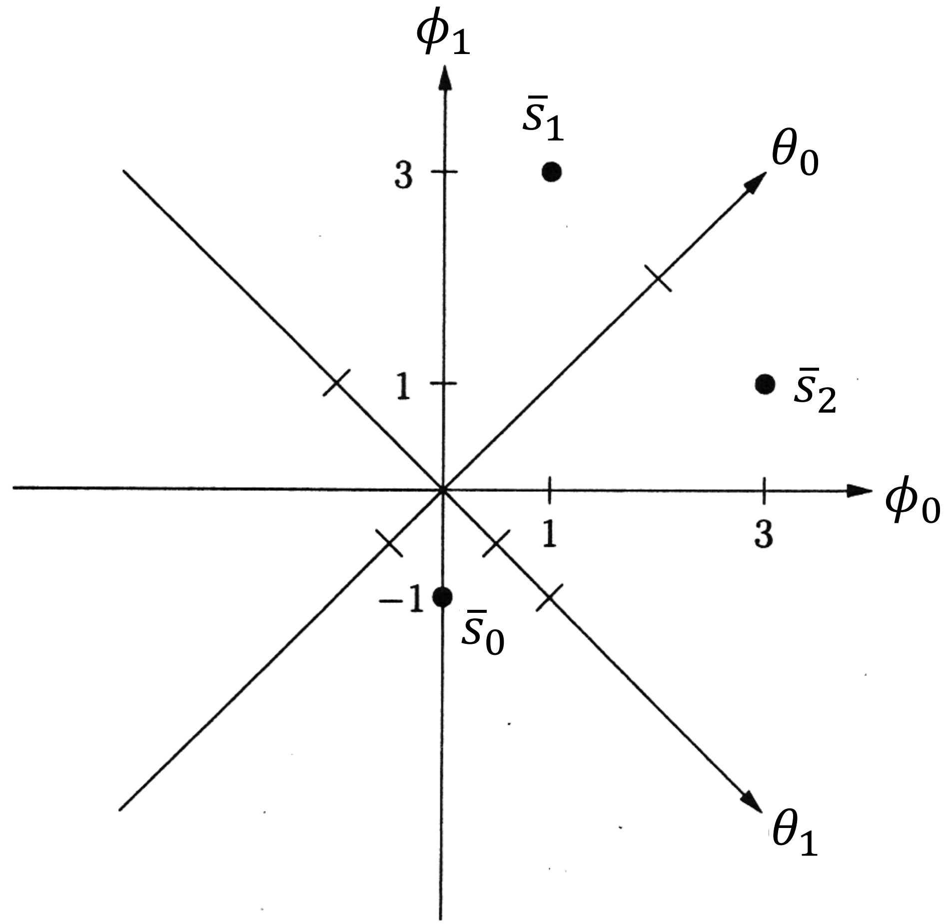
Example: Vector representation with respect to $\{\phi_0(t), \phi_1(t)\}$



Example: Vector representation with respect to $\{\theta_0(t), \theta_1(t)\}$



Example: Vector representation with respect to both bases



Geometric Interpretation

Signals: $a(t)$ and $b(t)$

ON-basis: $\{\phi_j(t)\}_{j=0}^{N-1}$

Vectors to represent signals: $\bar{a} = (a_0, a_1, \dots, a_{N-1})^T$

$\bar{b} = (b_0, b_1, \dots, b_{N-1})^T$

Inner products: $(a, b) \triangleq \int_{-\infty}^{\infty} a(t)b(t)dt$ $\qquad$ $\bar{a} \circ \bar{b} \triangleq \sum_{i=0}^{N-1} a_i b_i$

Claim: $(a, b) = \bar{a} \circ \bar{b}$

Geometric Interpretation

Claim: $(a, b) = \bar{a} \circ \bar{b}$

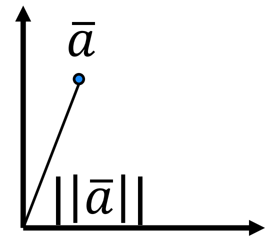
Proof:

$$(a,b) = \int_{-\infty}^{\infty} a(t)\, b(t)\, dt = \int_{-\infty}^{\infty} \sum_{i=0}^{N-1} a_i\, \phi_i(t) \sum_{j=0}^{N-1} b_j\, \phi_j(t)\, dt$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} a_i\, b_j \int_{-\infty}^{\infty} \phi_i(t)\, \phi_j(t)\, dt$$

$$= \sum_{i=0}^{N-1} a_i \sum_{j=0}^{N-1} b_j\, \delta_{i,j} = \sum_{i=0}^{N-1} a_i\, b_i = \overline{a} \circ \overline{b}.$$

Norms and Angles



Norms: $\|a\|$ of $a(t)$: $\|a\|^2 \triangleq (a, a)$ Signal energy.

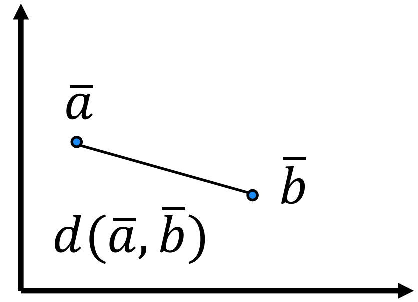
$\|\overline{a}\|$ of $\overline{a}$: $\|\overline{a}\|^2 \triangleq \overline{a} \circ \overline{a},$ Squared length.

Relation: $(a, b) = \overline{a} \circ \overline{b}.$ $\Rightarrow$ $\|a\| = \|\overline{a}\|$

From linear algebra: $\overline{a} \circ \overline{b} = \|\overline{a}\| \cdot \|\overline{b}\| \cdot \cos(\alpha)$

Angle between signals: $\cos(\alpha) = \dfrac{\overline{a} \circ \overline{b}}{\|\overline{a}\| \cdot \|\overline{b}\|} = \dfrac{(a, b)}{\|a\| \cdot \|b\|}$

Distances



Euclidean distances between vectors:

$$d^2(\bar{a},\bar{b}) = \|\bar{a}-\bar{b}\|^2 = \sum_{j=0}^{N-1} \left(a_j - b_j\right)^2$$

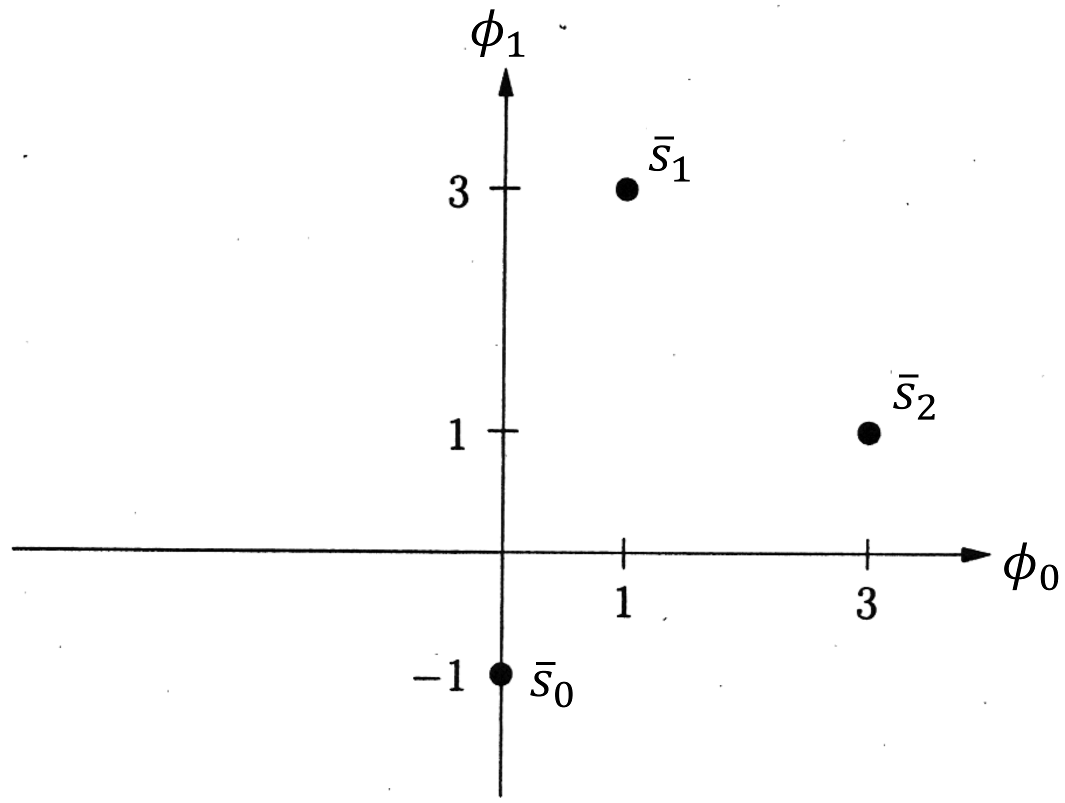
The Pythagorean relation in N dimensions.

Euclidean distances between signals:

$$d^2(\bar{a},\bar{b}) = \|a-b\|^2 = \int_{-\infty}^{\infty} \left(a(t) - b(t)\right)^2 dt$$

These computations are equivalent!

Example: Norms, Distances and Angles



Representing White Gaussian Noise 1(4)

Signal: $$s_i(t) = \sum_{j=0}^{N-1} s_{i,j}\, \underset{}{\overset{\text{ON}}{\phi_j}}(t), \quad i \in \{0, 1, \ldots, M-1\},$$

Noise: $W(t)$ WGN with PSD $R_W(f) = N_0/2$

Received: $X(t) = s_i(t) + W(t)$

Define: $W_j \triangleq (W, \phi_j)$

Gaussian variable	$\Leftarrow$	Gaussian process
Mean 0	$\Leftarrow$	Mean 0

Define: $X_j \triangleq (X, \phi_j)$

Gaussian variable	$\Leftarrow$	Gaussian process
Mean?	$\Leftarrow$	Mean $s_i(t)$

Also determine:

- Variances?
- Correlations?
- Independence?
- Full statistical description?

Representing White Gaussian Noise 2(4)

Correlate:
$$(s_i, \phi_k) = \int_0^T s_i(t)\,\phi_k(t)\,dt = \int_0^T \sum_{j=0}^{N-1} s_{i,j}\,\phi_j(t)\,\phi_k(t)\,dt$$
$$= \sum_{j=0}^{N-1} s_{i,j} \int_0^T \phi_j(t)\,\phi_k(t)\,dt = \sum_{j=0}^{N-1} s_{i,j}\,\delta_{j,k} = s_{i,k}$$

Correlate:
$$X_j \triangleq (X, \phi_j) \;= \big((s_i + W), \phi_j\big) = (s_i, \phi_j) + (W, \phi_j)$$
$$= s_{i,j} + W_j.$$

Result: X_j is Gaussian with mean $s_{i,j}$.

Next: Variance?

Time-limited basis functions to $[0, T)$

Representing White Gaussian Noise 3(4)

White noise: $r_W(u) = \frac{N_0}{2}\,\delta(u)$

Variance: $\sigma^2_{X_j} = \mathrm{E}\{(X_j - s_{i,j})^2\} = \mathrm{E}\{W_j^2\} = \mathrm{E}\left\{(W,\phi_j)^2\right\}$

$$= \mathrm{E}\left\{\int_0^T W(t)\,\phi_j(t)\,dt \int_0^T W(\tau)\,\phi_j(\tau)\,d\tau\right\} = \mathrm{E}\left\{\iint_0^T \phi_j(t)\,\phi_j(\tau)\,W(t)\,W(\tau)\,dt\,d\tau\right\}$$

$$= \iint_0^T \phi_j(t)\,\phi_j(\tau)\,E\left\{W(t)W(\tau)\right\}\,dt\,d\tau = \iint_0^T \phi_j(t)\,\phi_j(\tau)\,r_W(t-\tau)\,dt\,d\tau,$$

$$\underset{\substack{\uparrow \\ \text{White}}}{=} \frac{N_0}{2}\iint_0^T \phi_j(t)\,\phi_j(\tau)\,\delta(t-\tau)\,dt\,d\tau = \frac{N_0}{2}\int_0^T \phi_j^2(t)\,dt \underset{\substack{\uparrow \\ \text{ON}}}{=} \frac{N_0}{2}$$

Representing White Gaussian Noise 4(4)

Covariance $j \neq k$:

$$\lambda_{X_j,X_k} = \mathrm{E}\{(X_j - s_{i,j})(X_k - s_{i,k})\} = \mathrm{E}\{W_j W_k\} = \mathrm{E}\{(W,\phi_j)(W,\phi_k)\}$$

$$= \mathrm{E}\left\{\int_0^T W(t)\,\phi_j(t)\,dt \int_0^T W(\tau)\,\phi_k(\tau)\,d\tau\right\} = \mathrm{E}\left\{\iint_0^T \phi_j(t)\,\phi_k(\tau)\,W(t)\,W(\tau)\,dt\,d\tau\right\}$$

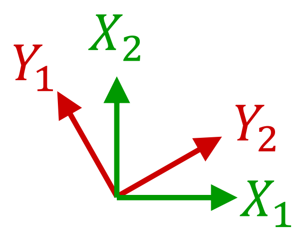
$$= \iint_0^T \phi_j(t)\,\phi_k(\tau)\,\mathrm{E}\{W(t)W(\tau)\}\,dt\,d\tau = \iint_0^T \phi_j(t)\,\phi_k(\tau)\,r_W(t-\tau)\,dt\,d\tau,$$

$$\underset{\text{White}}{=} \frac{N_0}{2}\iint_0^T \phi_j(t)\,\phi_k(\tau)\,\delta(t-\tau)\,dt\,d\tau = \frac{N_0}{2}\int_0^T \phi_j(t)\,\phi_k(t)\,dt \underset{\text{ON}}{=} 0.$$

Uncorrelated & jointly Gaussian $\Rightarrow$ Independent Gaussian

AWGN – Additive White Gaussian Noise

Orthogonal noise components
are statistically independent



X_1 and X_2 independent

Y_1 and Y_2 independent

Variance: $\sigma_{X_i}^2 = \sigma_{Y_i}^2 = R_W(f) = N_0/2$

PSD = Variance

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