# TSDT14 Signal Theory

## Lecture 7
## Analog Modulation

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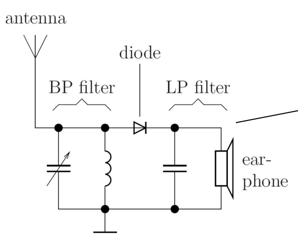
# Amplitude Modulation – Deterministic Case

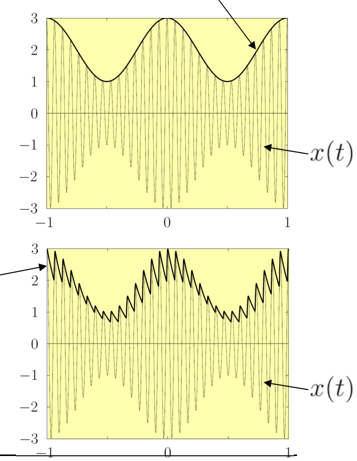
## Standard AM:

$$x(t) = A \cdot \big(C + m(t)\big) \cos(2\pi f_c t)$$

Crystal receiver, an envelope detector, first demodulator of standard AM:




Envelope, $A \cdot \big(C + m(t)\big)$



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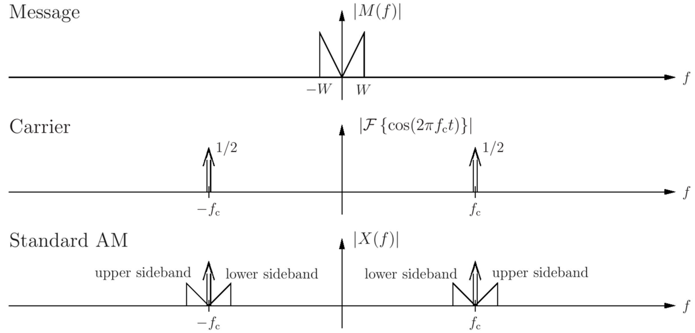
# Spectrum of Standard AM

$$x(t) = A \cdot \big(C + m(t)\big) \cos(2\pi f_c t)$$

$$X(f) = \mathcal{F}\big\{AC \cos(2\pi f_c t)\big\} + \mathcal{F}\big\{A\, m(t) \cos(2\pi f_c t)\big\}$$
$$= \frac{AC}{2}\big(\delta(f - f_c) + \delta(f + f_c)\big) + \frac{A}{2}\big(M(f - f_c) + M(f + f_c)\big),$$



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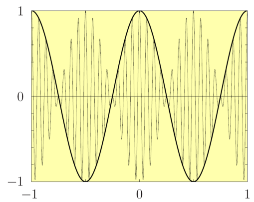
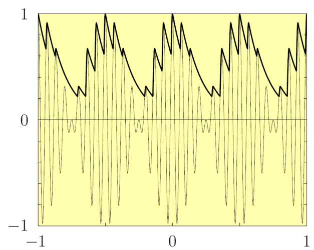
# AM-SC – Suppressed Carrier

AM-SC: $x(t) = A\, m(t) \cos(2\pi f_c t)$

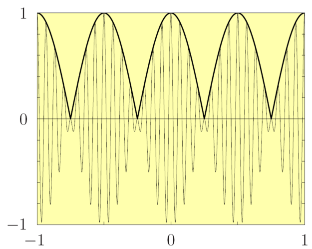


## Demodulating AM-SC with an envelope detector:

Crystal receiver output



Envelope



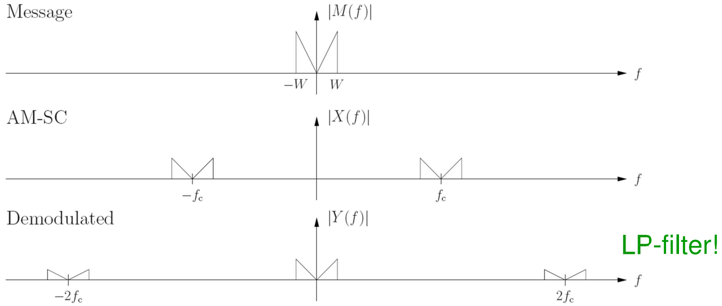
# Spectrum of AM-SC – and Demodulation

$$x(t) = A\,m(t)\cos(2\pi f_c t) \quad \Rightarrow \quad X(f) = \frac{A}{2}\big(M(f-f_c) + M(f+f_c)\big)$$

Coherent demodulation:

$$y(t) = 2\,x(t)\cos(2\pi f_c t) = 2A\,m(t)\cos^2(2\pi f_c t) = A\,m(t)\big(1+\cos(4\pi f_c t)\big),$$

$$Y(f) = A\big(X(f-f_c) + X(f+f_c)\big) = A\cdot M(f) + \frac{A}{2}\big(M(f-2f_c) + M(f+2f_c)\big).$$




LP-filter!

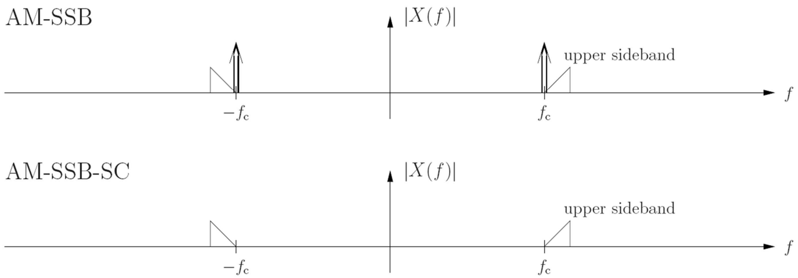


# AM-SSB – Single Side-Band

AM (SC) uses twice as much bandwidth as needed.

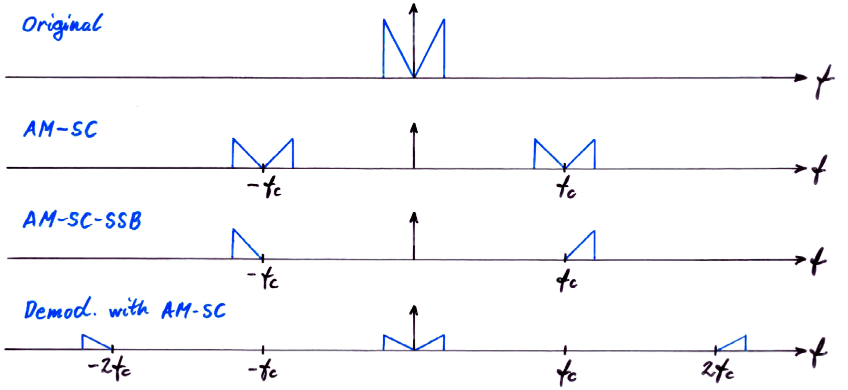
Each sideband contains all the information.

Filter out one of the sidebands.






# Demodulate SSB




LP-filter again!



# AM of Stochastic Processes 1(3)

As for deterministic signals: $X(t) = A\cdot\big(C + M(t)\big)\cos(2\pi f_c t)$

Question: Is $X(t)$ stationary in any sense if $M(t)$ is stationary?

ACF:

$$r_X(t, t+\tau) =$$

$$= \mathrm{E}\left\{A\big(C+M(t)\big)\cos(2\pi f_c t)\,A\big(C+M(t+\tau)\big)\cos\big(2\pi f_c(t+\tau)\big)\right\},$$

$$= A^2\mathrm{E}\left\{C^2 + C\big(M(t)+M(t+\tau)\big) + M(t)M(t+\tau)\right\}\cos(2\pi f_c t)\cos\big(2\pi f_c(t+\tau)\big)$$

$$= \frac{A^2}{2}\big(C^2 + 2Cm_M + r_M(\tau)\big)\Big(\cos\big(2\pi f_c(2t+\tau)\big) + \cos(2\pi f_c\tau)\Big),$$

Dependent on $t$. Non-stationary.

# AM of Stochastic Processes 2(3)

Adjust the situation: $X(t) = A \cdot \big(C + M(t)\big)\cos(2\pi f_c t + \Psi)$

Question: Is $X(t)$ stationary now?

$\Psi$: Unif. on $[0,2\pi)$. Indep of $X(t)$.

Mean: $m_X(t) = \mathrm{E}\{X(t)\} = \mathrm{E}\{A \cdot (C + M(t))\cos(2\pi f_c t + \Psi)\}$
$= \mathrm{E}\{A \cdot (C + M(t))\} \cdot \mathrm{E}\{\cos(2\pi f_c t + \Psi)\}$,

Note: $\mathrm{E}\{\cos(2\pi f_c t + \Psi)\} = \int_0^{2\pi} \frac{1}{2\pi}\cos(2\pi f_c t + \psi)\, d\psi = 0$,

Result: $m_X(t) = 0$

OK! Independent of $t$.
What about the ACF?



# AM of Stochastic Processes 3(3)

Still this situation: $X(t) = A \cdot \big(C + M(t)\big)\cos(2\pi f_c t + \Psi)$

ACF:

$$\begin{aligned} r_X(t,t+\tau) &= \mathrm{E}\left\{A\cdot\big(C+M(t)\big)\cos(2\pi f_c t+\Psi)\, A\cdot\big(C+M(t+\tau)\big)\cos\big(2\pi f_c(t+\tau)+\Psi\big)\right\}, \\ &= A^2\mathrm{E}\left\{C^2 + C\big(M(t)+M(t+\tau)\big) + M(t)M(t+\tau)\right\}\cdot \\ &\quad \cdot \mathrm{E}\left\{\cos(2\pi f_c t+\Psi)\cos\big(2\pi f_c(t+\tau)+\Psi\big)\right\}, \\ &= \frac{A^2}{2}\big(C^2 + 2Cm_M + r_M(\tau)\big)\Big(\mathrm{E}\left\{\cos\big(2\pi f_c(2t+\tau)+2\Psi\big)\right\} + \cos(2\pi f_c\tau)\Big), \end{aligned}$$

Note: $\mathrm{E}\left\{\cos\big(2\pi f_c(2t+\tau)+2\Psi\big)\right\} = \int_0^{2\pi}\frac{1}{2\pi}\cos\big(2\pi f_c(2t+\tau)+2\psi\big)\, d\psi = 0.$

Result: $r_X(t,t+\tau) = \frac{A^2}{2}\big(C^2 + 2Cm_M + r_M(\tau)\big)\cos(2\pi f_c\tau)$,

OK! Independent of $t$. Stationary in the wide sense.



# PSD of AM

Often, we have: $m_M = 0.$

ACF: $r_X(\tau) = \frac{A^2}{2}\big(C^2 + r_M(\tau)\big)\cos(2\pi f_c\tau)$

PSD: $R_X(f) = \frac{A^2C^2}{4}\big(\delta(f+f_c)+\delta(f-f_c)\big) + \frac{A^2}{4}\big(R_M(f+f_c)+R_M(f-f_c)\big)$

Power: $P_X = r_X(0) = \frac{A^2}{2}\big(C^2 + r_M(0)\big)\cos(0) = \frac{A^2}{2}\big(C^2 + r_M(0)\big) = \frac{A^2}{2}\big(C^2 + P_M\big)$

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For AM-SC:

$$r_X(\tau) = \frac{A^2}{2} r_M(\tau)\cos(2\pi f_c\tau),$$
$$R_X(f) = \frac{A^2}{4}\big(R_M(f+f_c)+R_M(f-f_c)\big),$$
$$P_X = \frac{A^2}{2}P_M.$$



# Coherent Demodulation of AM-SC

Coherent demodulation: $Y(t) = 2X(t)\cos(2\pi f_c t + \Psi)$

Same $\Psi$ as for modulation.

Demodulated signal:

$$Y(t) = 2A\,M(t)\cos^2(2\pi f_c t+\Psi) = A\,M(t)\big(1+\cos(4\pi f_c t + 2\Psi)\big),$$

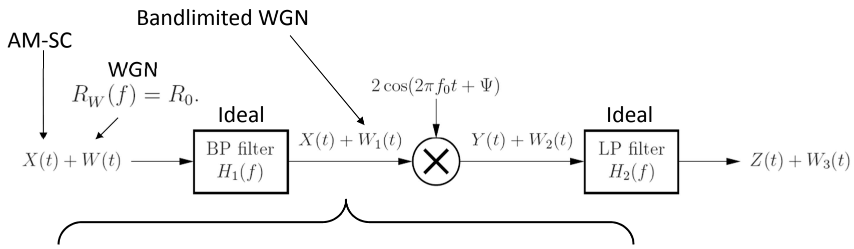
Result:

$$r_Y(\tau) = A^2 r_M(\tau) + \frac{A^2}{2} r_M(\tau)\cos(4\pi f_c\tau),$$
$$R_Y(f) = A^2R_M(f) + \frac{A^2}{4}\big(R_M(f+2f_c)+R_M(f-2f_c)\big),$$
$$P_Y = \frac{3A^2}{2}P_M.$$

After ideal LP-filter:

$r_Z(\tau) = A^2 r_M(\tau)$, $R_Z(f) = A^2R_M(f)$, $P_Z = A^2P_M$.

# Noise Analysis of AM-SC 1(4)




$$H_1(f) = \begin{cases} 1, & ||f|-f_c| < B, \\ 0, & \text{elsewhere.} \end{cases}$$

$$R_{W_1}(f) = |H_1(f)|^2 R_W(f) = \begin{cases} R_0, & ||f|-f_c| < B, \\ 0, & \text{elsewhere,} \end{cases}$$

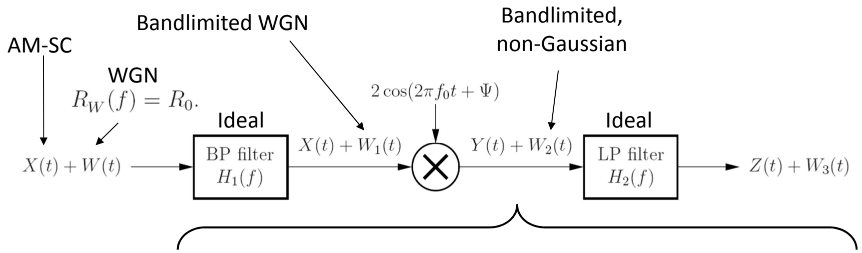
$$P_{W_1} = \int_{-\infty}^{\infty} R_{W_1}(f)\,df = 2\int_{f_c-B}^{f_c+B} R_0\,df = 4BR_0.$$

**Input SNR**

$$\frac{P_X}{P_{W_1}} = \frac{A^2P_M/2}{4BR_0} = \frac{A^2P_M}{8BR_0}$$



# Noise Analysis of AM-SC 2(4)

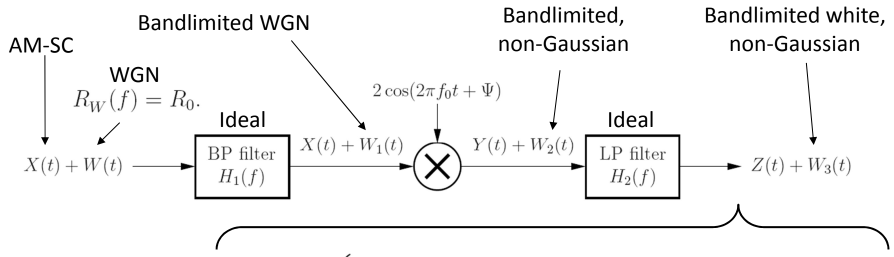



$$W_2(t) = 2W_1(t)\cos(2\pi f_c t + \Psi)$$

$$R_{W_2}(f) = R_{W_1}(f+f_c) + R_{W_1}(f-f_c) = \begin{cases} 2R_0, & |f| < B, \\ R_0, & ||f|-2f_c| < B, \\ 0, & \text{elsewhere.} \end{cases}$$



# Noise Analysis of AM-SC 3(4)




$$H_2(f) = \begin{cases} 1, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$$

**Output SNR**

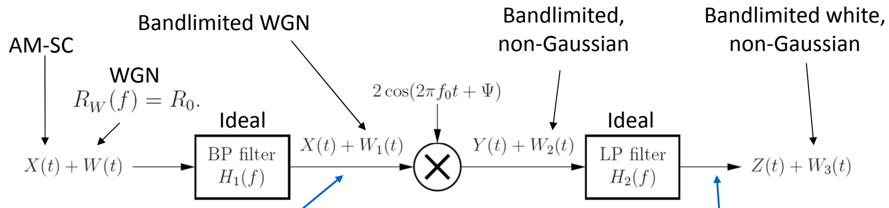
$$\frac{P_Z}{P_{W_3}} = \frac{A^2P_M}{4BR_0}$$

$$R_{W_3}(f) = |H_2(f)|^2 R_{W_2}(f) = \begin{cases} 2R_0, & |f| < B, \\ 0, & \text{elsewhere,} \end{cases}$$

$$P_{W_3} = \int_{-\infty}^{\infty} R_{W_3}(f)\,df = \int_{-B}^{B} 2R_0\,df = 4BR_0.$$



# Noise Analysis of AM-SC 4(4)




**Input SNR**

$$\frac{P_X}{P_{W_1}} = \frac{A^2P_M}{8BR_0}$$

**SNR Relation**

$$\frac{P_Z}{P_{W_3}} = 2\,\frac{P_X}{P_{W_1}}$$

**Output SNR**

$$\frac{P_Z}{P_{W_3}} = \frac{A^2P_M}{4BR_0}$$

Twice the bandwidth – Twice the SNR.

# Angle Modulation

- The major modulation techniques used in radio broadcasts today are examples of angle modulation.
  - FM – Frequency Modulation
  - PM – Phase Modulation
- Nonlinear modulation techniques.
- Complicated to analyze.
- Still fairly simple demodulation.
- Less sensitive to noise than AM.



# Angle Modulation – Modulation Indices

Angle modulation: $x(t) = A \cdot \cos\left(2\pi f_c t + \underbrace{\phi\{m(t)\}}_{\text{Momentary phase}}\right)$

Phase deviation: $\phi_d(t) = \phi\{m(t)\}$, for mean 0.

Phase modulation index: $\mu_p = \phi_{d,\max} = \max|\phi_d(t)|$

Momentary frequency: $f_{mom}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt}(2\pi f_c t + \phi\{m(t)\}) = f_c + \frac{1}{2\pi} \cdot \frac{d}{dt}\phi\{m(t)\}$

Frequency deviation: $f_d(t) = f_{mom}(t) - f_c = \frac{1}{2\pi} \cdot \frac{d}{dt}\phi\{m(t)\}$

Frequency modulation idx: $\mu_f = \frac{f_{d,\max}}{B}$ $\quad f_{d,\max} = \max|f_d(t)|$

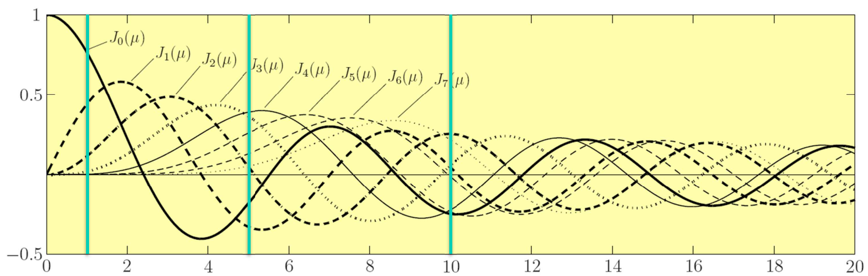


# Spectrum of Angle Modulation 1(2)

Example: $x(t) = A \cdot \cos\left(2\pi f_c t + \mu \sin(2\pi f_m t)\right) = \sum_{n=-\infty}^{\infty} A \cdot J_n(\mu) \cos\left(2\pi(f_c + n f_m)t\right)$

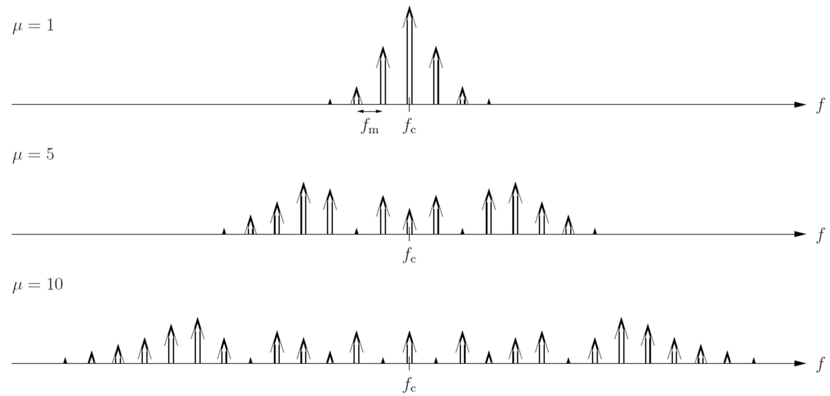
Spectrum: $X(f) = \sum_{n=-\infty}^{\infty} \frac{A \cdot J_n(\mu)}{2}\Big(\delta(f + f_c + n f_m) + \delta(f - f_c - n f_m)\Big).$

Bessel functions of the first kind of order $n$: $J_n(\mu) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{\mu}{2}\right)^{n+2k}$






# Spectrum of Angle Modulation 2(2)




Carsons rule: Bandwidth ≈ $2(\mu+1)f_m = 2\left(1 + \frac{1}{\mu}\right) f_{d,\max}$.

# PM – Phase Modulation

Momentary phase: $\phi\{m(t)\} = a \cdot m(t),$

Signal: $x(t) = A \cdot \cos\big(2\pi f_c t + a \cdot m(t)\big).$

Momentary frequency: $f_{\mathrm{mom}}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt}\big(2\pi f_c t + a \cdot m(t)\big) = f_c + \frac{a}{2\pi} \cdot \frac{d}{dt} m(t),$

Frequency deviation: $f_{\mathrm{d}}(t) = \frac{a}{2\pi} \cdot \frac{d}{dt} m(t).$

Peak frequency dev.: $f_{\mathrm{d,max}} = \frac{a}{2\pi} \cdot \max\left|\frac{d}{dt} m(t)\right|$

Frequency mod. idx: $\mu_{\mathrm{f}} = \frac{a}{2\pi B} \cdot \max\left|\frac{d}{dt} m(t)\right|$



# FM – Frequency Modulation

Momentary phase: $\phi\{m(t)\} = a \int m(t)\,dt,$ $\qquad \phi\{m(t)\} = a \int_{t_0}^{t} m(\tau)\,d\tau,$

Signal: $x(t) = \cos\left(2\pi f_c t + a \int m(t)\,dt\right)$

Momentary frequency: $f_{\mathrm{mom}}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt}\left(2\pi f_c t + a \int m(t)\,dt\right) = f_c + \frac{a}{2\pi} \cdot m(t).$

Frequency deviation: $f_{\mathrm{d}}(t) = \frac{a}{2\pi} \cdot m(t).$

Peak frequency dev.: $f_{\mathrm{d,max}} = \frac{a}{2\pi} \cdot \max|m(t)|$

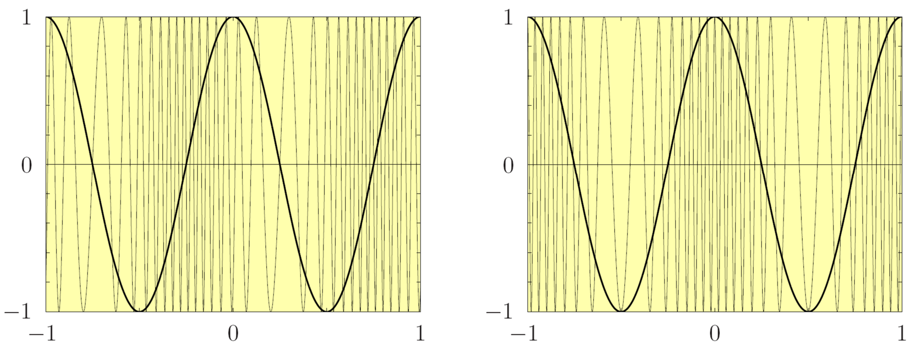
Frequency mod. idx: $\mu_{\mathrm{f}} = \frac{a}{2\pi B} \cdot \max|m(t)|$



# Angle Modulation in the Time Domain

Phase Modulation

Frequency Modulation






# Demodulating Phase Modulation

Sent signal: $x(t) = A \cdot \cos\big(2\pi f_c t + \phi\{m(t)\}\big)$

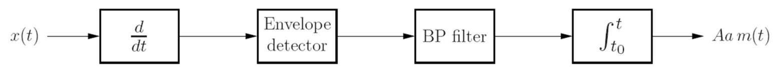
Derivative: $\frac{d}{dt} x(t) = -A\left(2\pi f_c + \frac{d}{dt}\phi\{m(t)\}\right)\sin\big(2\pi f_c t + \phi\{m(t)\}\big)$

Envelope detector gives us: $A\left(2\pi f_c + \frac{d}{dt}\phi\{m(t)\}\right)$

BP or HP filter gives us: $A\frac{d}{dt}\phi\{m(t)\}$

For PM: $\phi\{m(t)\} = am(t).$

Integrate: $\int_{t_0}^{t} A\frac{d}{d\tau}\phi\{m(\tau)\}\,d\tau = Aa\,\big(m(t) - m(t_0)\big)$ Filter!



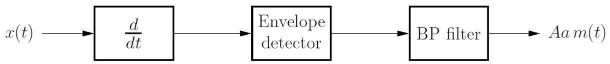
# Demodulating Frequency Modulation

As for PM: Derivative, envelope detection and HP filter gives us:

$$A\tfrac{d}{dt}\phi\{m(t)\}$$

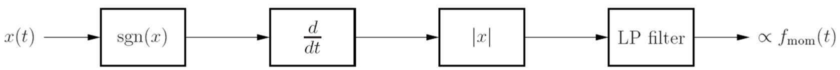
For FM: $\phi\{m(t)\} = a\int m(t)\,dt.$

Then: $A\dfrac{d}{dt}\phi\{m(t)\} = A\dfrac{d}{dt}a\displaystyle\int m(t)\,dt = Aa\,m(t)$ Done!





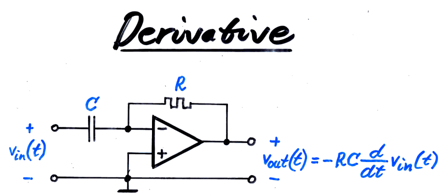

# Alternative Demodulation

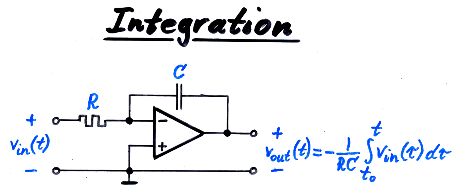


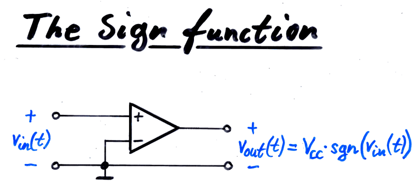

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ \text{-}1, & x < 0. \end{cases}$$



# Building Blocks







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