TSKS01 Digital Communication Lecture 5

Digital modulation: Bounds, approximations, standard constellations

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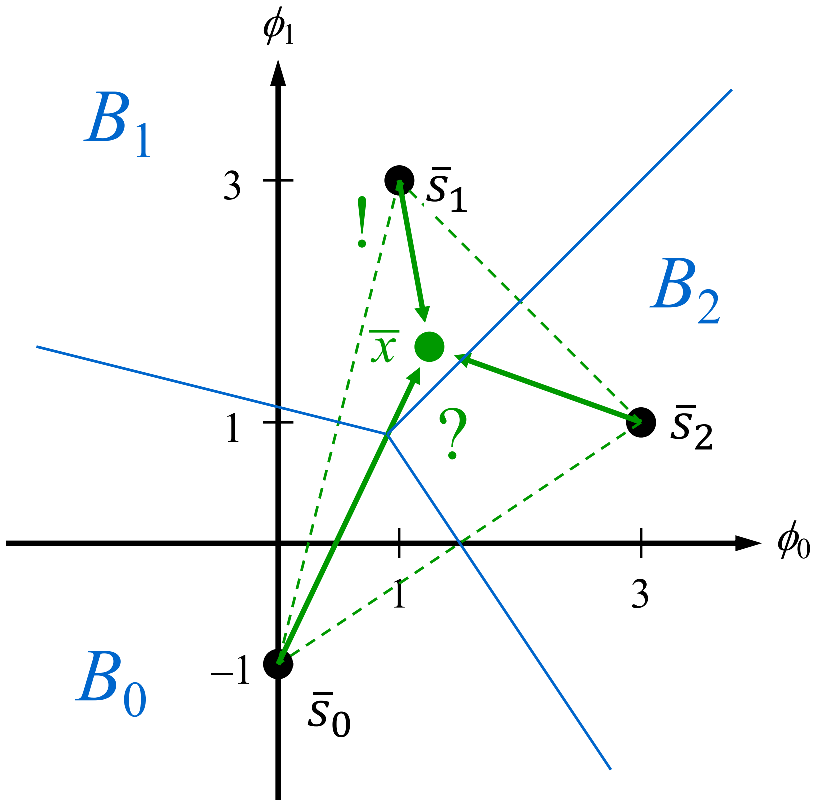
Recall : ML Decision Regions



Result:

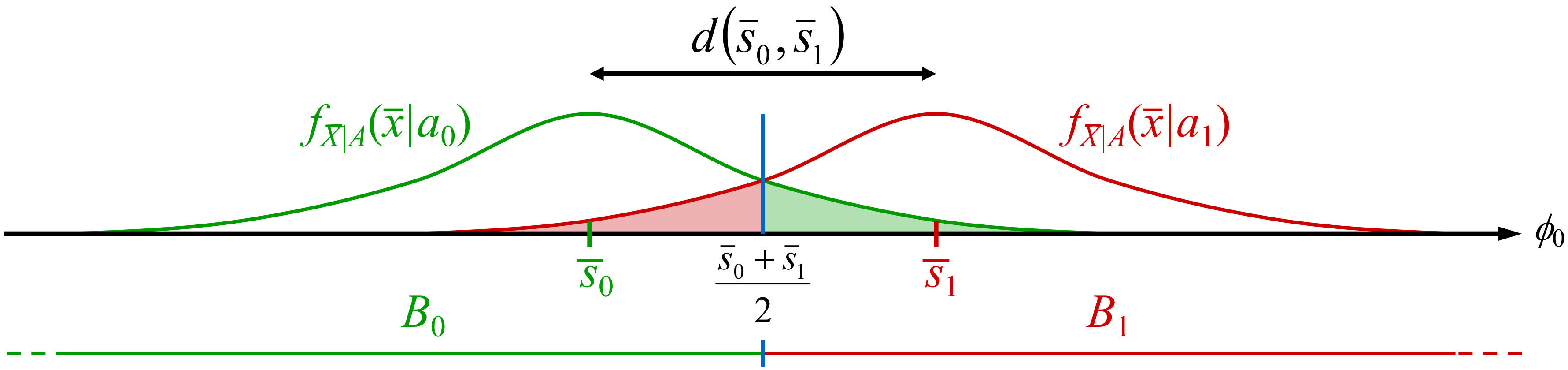
Decision regions consist of all points closest to a signal point.

Notation:

B_i is the decision region of the signal vector $\bar{s}_i$.

Detect $\bar{x}$ as being the nearest signal.

Recall: Two signals in $N = 1$ Dimension



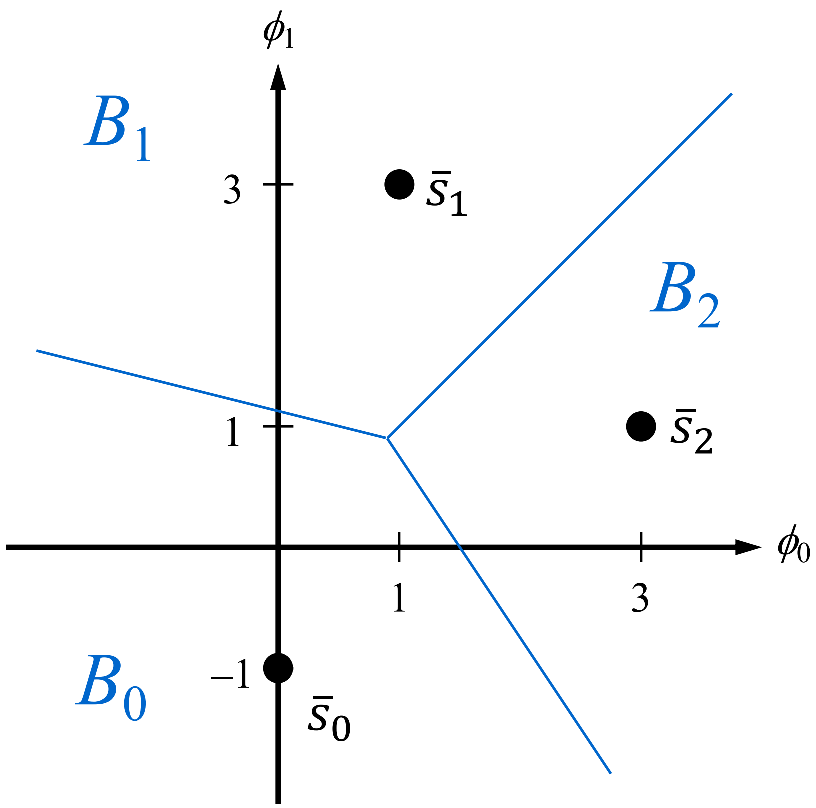
Detection:

$$\bar{X} = \begin{cases} a_0 \text{ if } \bar{x} \in B_0 \\ a_1 \text{ if } \bar{x} \in B_1 \end{cases}$$

Probability of error:

$$\mathrm{P}_e = \Pr\{A \neq \hat{A}\} = \frac{\Pr\{\bar{X} \in B_1 | A = a_0\} + \Pr\{\bar{X} \in B_0 | A = a_1\}}{2} = Q\left(\frac{d(\bar{s}_0, \bar{s}_1)}{\sqrt{2N_0}}\right)$$

Back to M Signals in N Dimensions



Interprete $\bar{x}$ as the nearest signal.

We had:

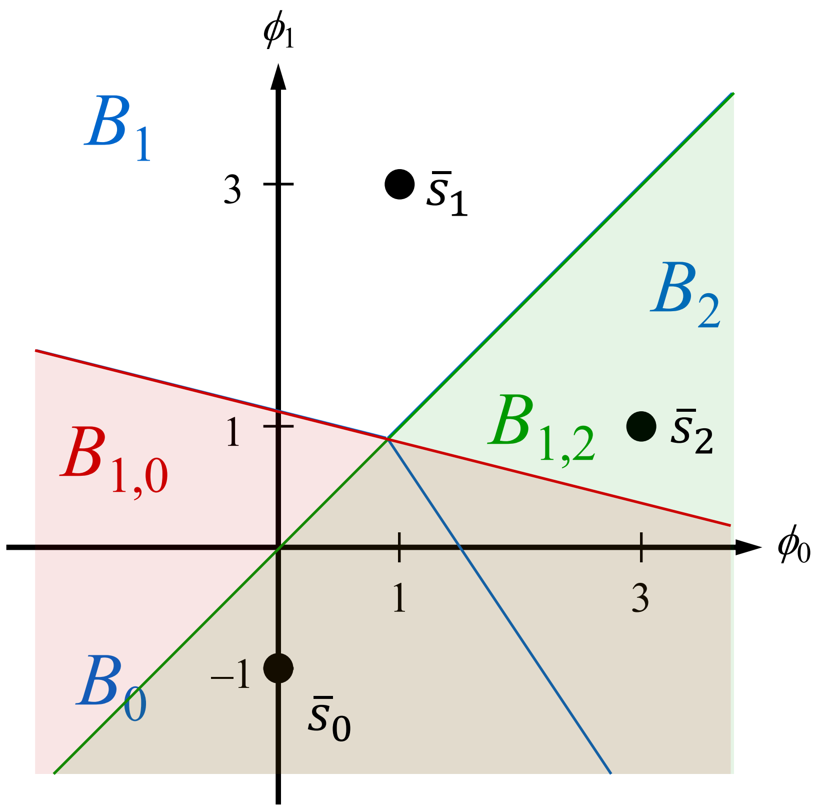
$$P_e = \frac{1}{M}\sum_{i=0}^{M-1}\Pr\left\{\bar{X} \notin B_i \mid A = a_i\right\}$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i}\Pr\left\{\bar{X} \in B_j \mid A = a_i\right\}$$

Hard to calculate!

Could we take this to the simpler case in one dimension?

The Union Bound



Interprete $\bar{x}$ as the nearest signal.

Upper bound by overestimating the decision regions.

We had:

$$P_e = \frac{1}{M} \sum_{i=0}^{M-1} \sum_{j \neq i} \Pr\left\{ \bar{X} \in B_j \mid A = a_i \right\}$$

Define overestimated regions:

$$B_{i,j} = \left\{ \bar{x} : d(\bar{x}, \bar{s}_j) < d(\bar{x}, \bar{s}_i) \right\}$$

Overestimated error probability:

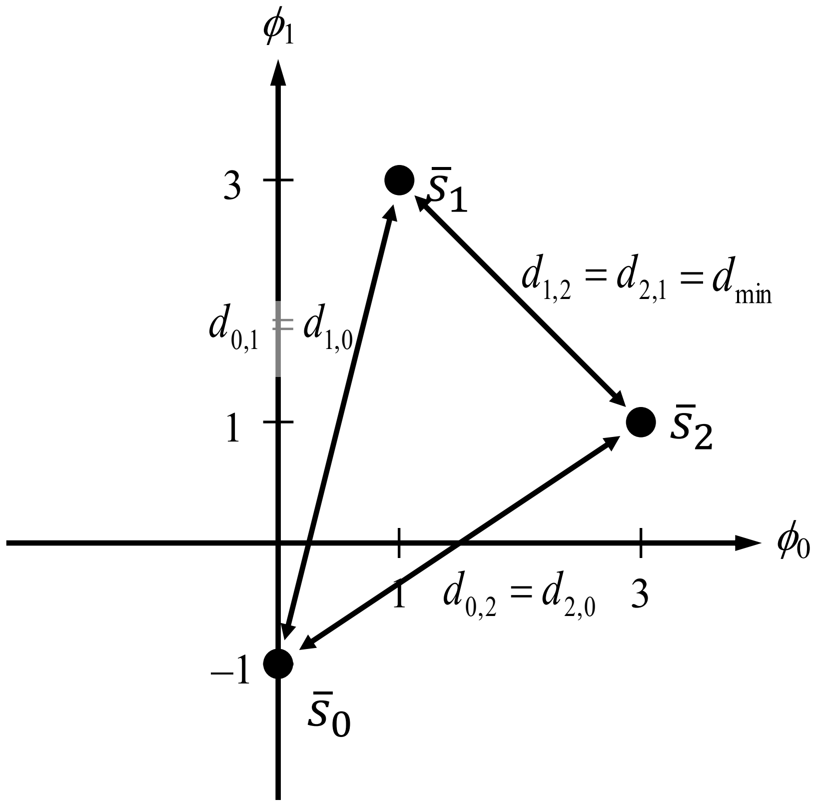
$$P_e \leq \frac{1}{M} \sum_{i=0}^{M-1} \sum_{j \neq i} \Pr\left\{ \bar{X} \in B_{i,j} \mid A = a_i \right\}$$

In the one-dimensional case:

$$P_e \leq \frac{1}{M} \sum_{i=0}^{M-1} \sum_{j \neq i} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}} \right)$$

Distances: $d_{i,j} = d(\bar{s}_i, \bar{s}_j)$

The Nearest Neighbour Approximation



Interprete $\bar{x}$ as the nearest signal.

We had the union bound:

$$P_e \leq \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

Dominated by the smallest distance:

$$d_{\min} = \min_{i\neq j} \; d_{i,j}$$

$$n_i = \# \, j : d_{i,j} = d_{\min}$$

Nearest neighbour approximation:

$$P_e \approx \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j:d_{i,j}=d_{\min}} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} n_i Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Comments

$$E_{\text{avg}} = \frac{1}{M}\sum_{i=0}^{M-1} E_i\,, \qquad E_i = \int_{-\infty}^{\infty} s_i^2(t)dt < \infty$$

At high SNR E_{avg}/N_0:

Both the union bound on (and the nearest neighbour of) the error probability are close to the real error probability.

Alternative upper bound:

As the union bound, but only consider pairs of points whose decision regions share a common border.

Alternative approximation:

As the nearest neighbour approximation, but consider the two or three smallest distances.

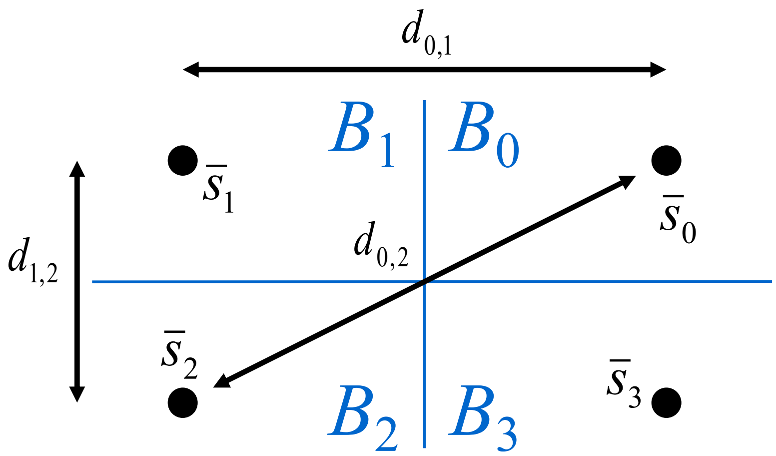
Very simple approximation:

$$P_e \approx Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Very simple upper bound:

$$P_e \leq (M-1)\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Special Case: Orthogonal Decision Borders



Define notation: $q_{\mathrm{i,j}} = Q\left(\dfrac{d_{i,j}}{\sqrt{2N_0}}\right)$

Pythagoras: $d_{0,2}^2 = d_{0,1}^2 + d_{1,2}^2$

UB: $P_{\mathrm{e}} \leq q_{0,1} + q_{1,2} + q_{0,2}$

Alternative bound: $P_{\mathrm{e}} \leq q_{0,1} + q_{1,2}$

NN: $P_{\mathrm{e}} \approx q_{1,2}$

Alternative approx.: $P_{\mathrm{e}} \approx q_{0,1} + q_{1,2}$

Exact: $P_{\mathrm{e}} = q_{0,1} + q_{1,2} - q_{0,1}q_{1,2}$

Lower bound: $P_{\mathrm{e}} > q_{1,2}$

(orthogonal noise components are independent)

Designing a Constellation

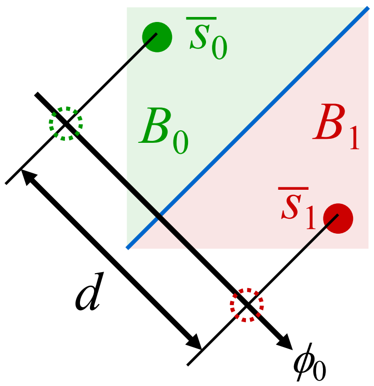
- Practical limitations
  - Energy per symbol – average or maximum
  - Energy per bit – average or maximum

- Recall definitions:

$$E_i = \int_0^T s_i^2(t)dt = \|\bar{s}_i\|^2$$

  - Average per symbol: $E_{\text{avg}} = \frac{1}{M}\sum_{i=0}^{M-1} E_i$
  - Maximum per symbol: $E_{\text{max}} = \max_{0 \le i \le M-1} E_i$
  - Energy per bit: Divide E by $\log_2(M)$

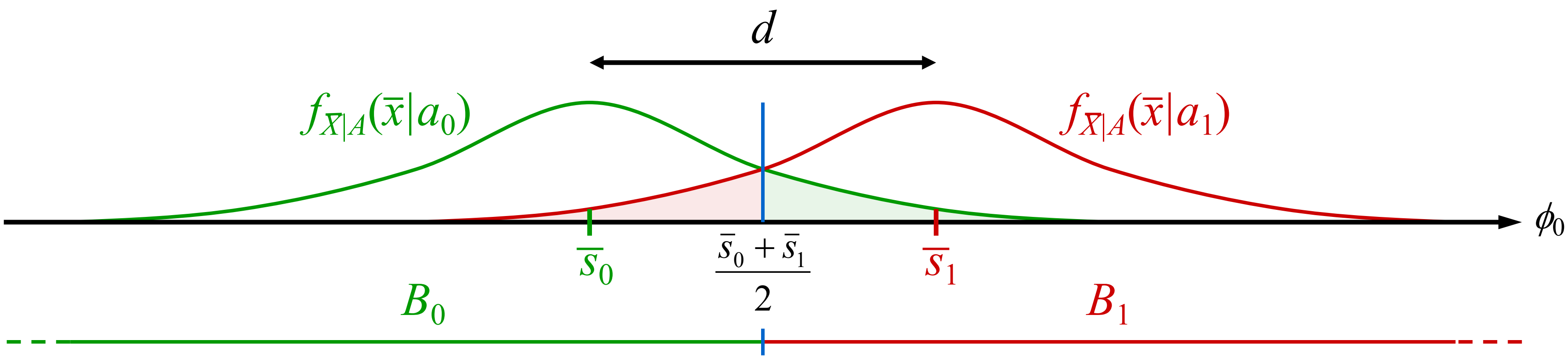
Binary Constellations ($M = 2$)



$$P_e = Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

$$E_{avg} = \frac{E_0 + E_1}{2} = \frac{\|\bar{s}_0\|^2 + \|\bar{s}_1\|^2}{2}$$

$$E_{max} = \max(\|\bar{s}_0\|^2, \|\bar{s}_1\|^2)$$



On-Off Keying (OOK)

Basis:

$$\phi(t) = \sqrt{2/T}\cos(2\pi f_c t), \quad 0 \leq t < T$$

Signals:

$$s_0(t) = 0, \quad s_1(t) = A \cdot \phi(t)$$

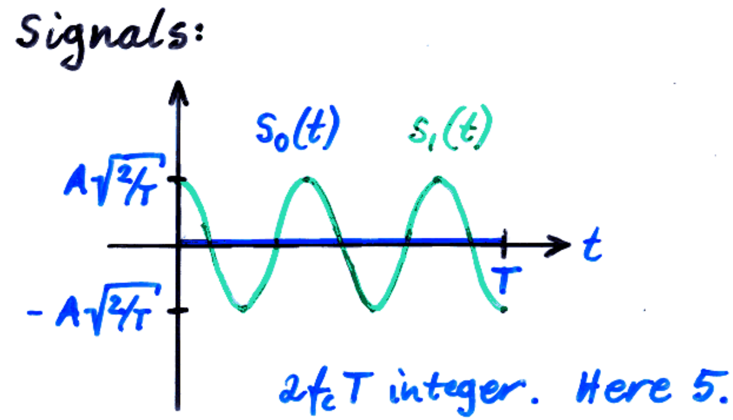
Energies:

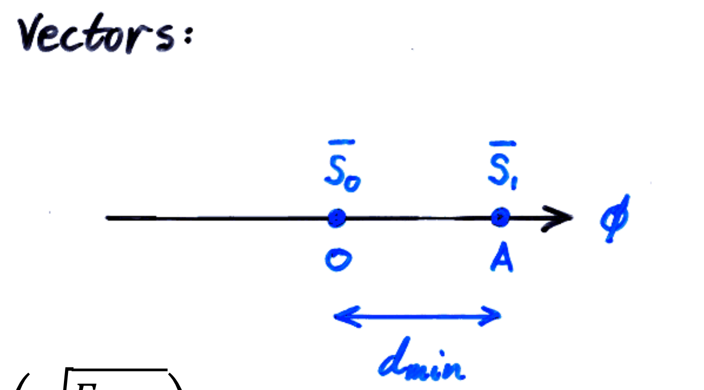
$$E_0 = 0, \; E_1 = A^2$$

$$E_{\text{avg}} = \frac{0+A^2}{2} = \frac{A^2}{2}, \quad E_{\max} = E_1 = A^2$$

Error probability (AWGN):

$$P_e = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = Q\left(\frac{A}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_{avg}}{N_0}}\right) = Q\left(\sqrt{\frac{E_{\max}}{2N_0}}\right)$$





Binary Phase-Shift Keying (BPSK)

Basis:

$$\phi(t) = \sqrt{2/T}\cos(2\pi f_c t), \quad 0 \le t < T$$

Signals:

$$s_0(t) = +A \cdot \phi(t)$$

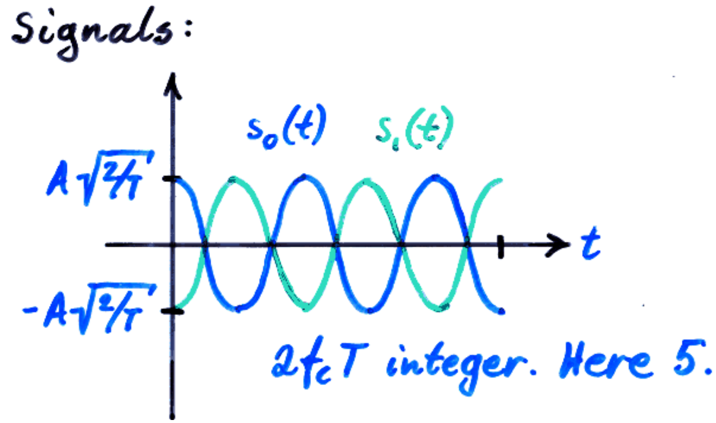
$$s_1(t) = -A \cdot \phi(t)$$

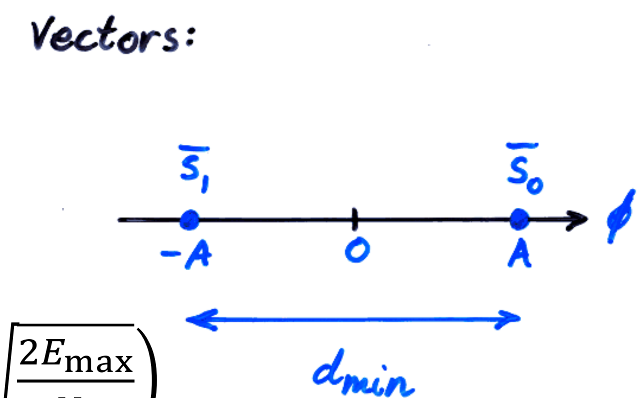
Energies:

$$E_0 = E_1 = E_{\text{avg}} = E_{\text{max}} = A^2$$

Error probability (AWGN):

$$P_e = Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right) = Q\left(\frac{2A}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2E_{\text{avg}}}{N_0}}\right) = Q\left(\sqrt{\frac{2E_{\text{max}}}{N_0}}\right)$$





Binary Frequency-Shift Keying (BFSK)

Basis:

$$\phi_0(t) = \sqrt{2/T}\cos(2\pi f_0 t), \quad 0 \le t < T$$

$$\phi_1(t) = \sqrt{2/T}\cos(2\pi f_1 t), \quad 0 \le t < T$$

Signals:

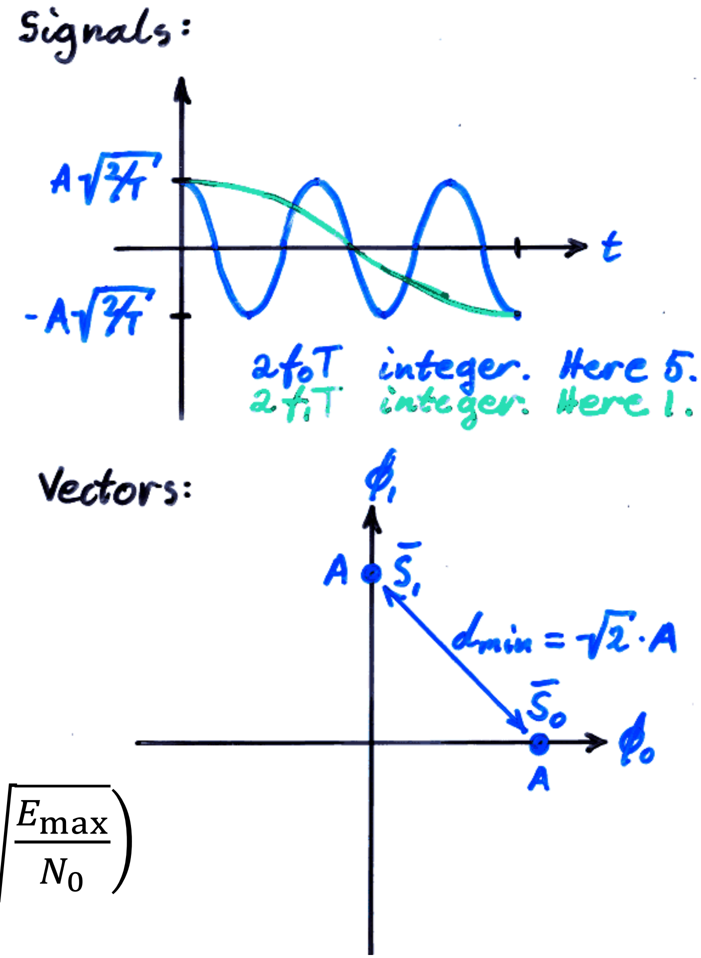
$$s_0(t) = A \cdot \phi_0(t), \quad s_1(t) = A \cdot \phi_1(t)$$

Energies:

$$E_0 = E_1 = E_{\text{avg}} = E_{\max} = A^2$$

Error probability (AWGN):

$$P_e = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = Q\left(\frac{A}{\sqrt{N_0}}\right) = Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right) = Q\left(\sqrt{\frac{E_{\max}}{N_0}}\right)$$



Comparison of Binary Constellations

- Fair comparison: Same average energy E_{avg}!
  - Error probability
    - Q-function is decreasing: Larger argument is better
    - OOK: $P_e = Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right)$
    - BPSK: $P_e = Q\left(\sqrt{\frac{2E_{\text{avg}}}{N_0}}\right)$ ← Lowest error probability ($\times 2 = 3$ dB better SNR)
    - BFSK: $P_e = Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right)$

BPSK uses phase difference ($+A/-A$) to maximize $d_{\min}$ for given energy!

Drawback: Phase coherence (e.g., OOK only needs to compute $||\bar{x}||$)

Non-Binary Constellations

Symbol error probability: P_e

Bit error probability: P_b

Maximum symbol energy: E_max

Average symbol energy: E_avg

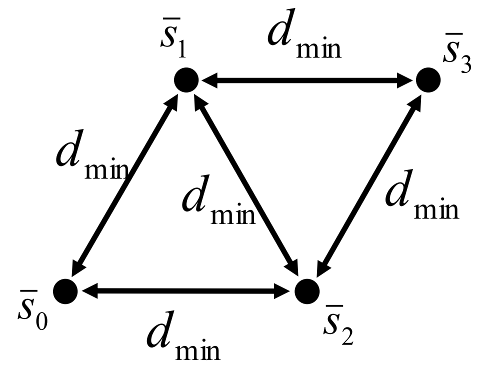
Average bit energy: E_b

Number of bits: k

Number of signals: $M = 2^k$

Relation: $E_\mathrm{avg} = kE_\mathrm{b}$

Example:



Number of nearest neighbours:

$$n_0 = n_3 = 2$$

Nearest neighbour approximation:

$$P_\mathrm{e} \approx \frac{1}{M}\sum_{i=0}^{M-1} n_i Q\left(\frac{d_\mathrm{min}}{\sqrt{2N_0}}\right)$$

$$= \frac{1}{4}(2+3+3+2)\cdot Q(\cdots) = \frac{5}{2}Q(\cdots)$$

Error Probabilities

Number of bit positions where s_i and s_j differ.

Exact:

$$P_e = \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i}\Pr\left\{\overline{X}\in B_j \mid A=a_i\right\}$$

$$P_b = \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i}\frac{N_{i,j}}{k}\Pr\left\{\overline{X}\in B_j \mid A=a_i\right\}$$

Union bound:

$$P_e \leq \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i}Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

$$P_b \leq \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i}\frac{N_{i,j}}{k}Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

Nearest neighbour approx:

$$P_e \approx \frac{1}{M}\sum_{i=0}^{M-1}\underbrace{\sum_{j:\,d_{ij}=d_{\min}}Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)}_{n_i Q(\cdots)}$$

$$P_b \approx \underbrace{\frac{1}{M}\sum_{i=0}^{M-1}\sum_{j:\,d_{ij}=d_{\min}}\frac{N_{i,j}}{k}}Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Approx. of average fraction of bits that differ between a signal and a neighbour, counting only nearest neighbours.

Gray Codes

Subsequent patterns differ in only one bit position.

Makes nearest neighbours differ in only one bit position: $P_b \approx P_e/k$.

One bit

0
1

Two bits

0	0
0	1
1	1
1	0

First two rows: One bit. Last two rows: One bit reflected.

Three bits

0	0	0
0	0	1
0	1	1
0	1	0
1	1	0
1	1	1
1	0	1
1	0	0

First four rows: Two bits. Last four rows: Two bits reflected.

In general, n bits:

0	$0\ 0 \cdots 0$
$\vdots$	$\ddots$
0	$1\ 0 \cdots 0$
1	$1\ 0 \cdots 0$
$\vdots$	$\cdot^{\cdot^{\cdot}}$
1	$0\ 0 \cdots 0$

Upper half: n-1 bits. Lower half: n-1 bits reflected.

Amplitude-Shift Keying (ASK)

Basis:

$$\phi(t) = \sqrt{2/T}\cos(2\pi f_c t), \quad 0 \le t < T$$

Signals ($M = 4$):

$$s_1(t) = -s_2(t) = A \cdot \phi(t)$$

$$s_0(t) = -s_3(t) = 3A \cdot \phi(t)$$

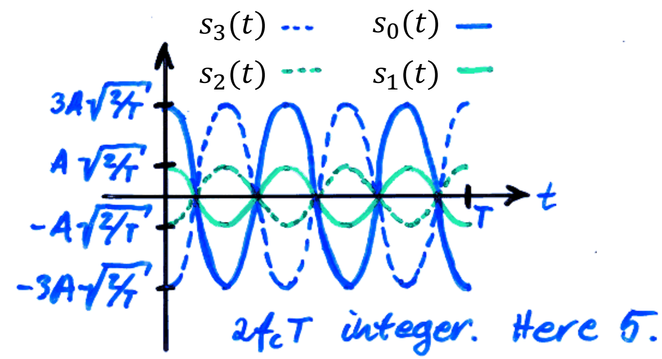
Energies:

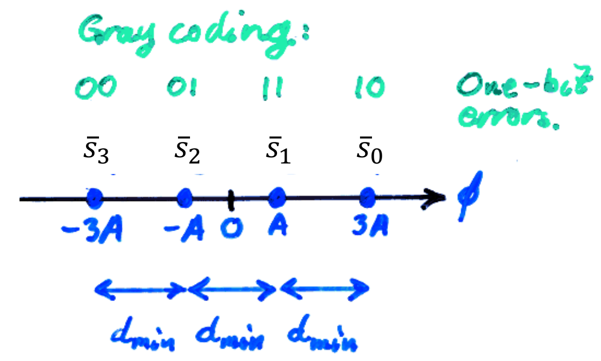
$$E_{\text{avg}} = 5A^2,\ E_{\max} = 9A^2,\ E_b = 5A^2/2$$

Error probability (AWGN):

$$P_e \approx \frac{1+2+2+1}{4} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = \frac{3}{2} Q\left(\frac{2A}{\sqrt{2N_0}}\right) = \frac{3}{2} Q\left(\sqrt{\frac{2E_{\text{avg}}}{5N_0}}\right) = \frac{3}{2} Q\left(\sqrt{\frac{2E_{\max}}{9N_0}}\right)$$

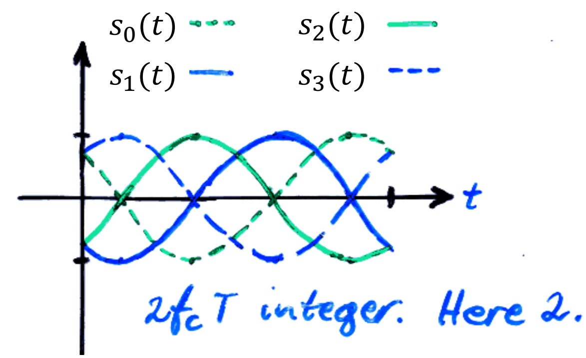
$$P_b \approx \frac{P_e}{2} = \frac{3}{4} Q\left(\sqrt{\frac{2E_{\text{avg}}}{5N_0}}\right) = \frac{3}{4} Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$





Quadruple Phase-Shift Keying (QPSK/4-PSK)

Basis:



$$\phi_0(t) = \sqrt{2/T}\cos(2\pi f_c t), \quad 0 \leq t < T$$

$$\phi_1(t) = \sqrt{2/T}\sin(2\pi f_c t), \quad 0 \leq t < T$$

Signals ($i = 0,1,2,3$):

$$s_i(t) = A\cos\left(\frac{(2i+1)\pi}{4}\right)\phi_0(t) - A\sin\left(\frac{(2i+1)\pi}{4}\right)\phi_1(t)$$

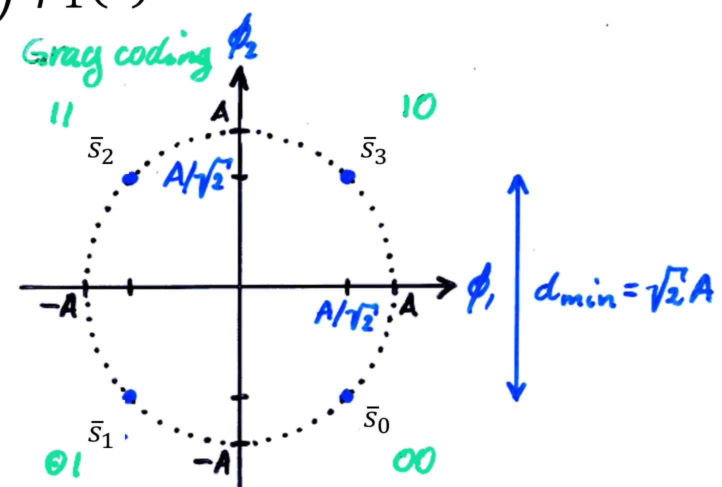
Energies:

$$E_i = E_{\text{avg}} = E_{\max} = A^2,\ E_b = A^2/2$$

Error probability (AWGN):

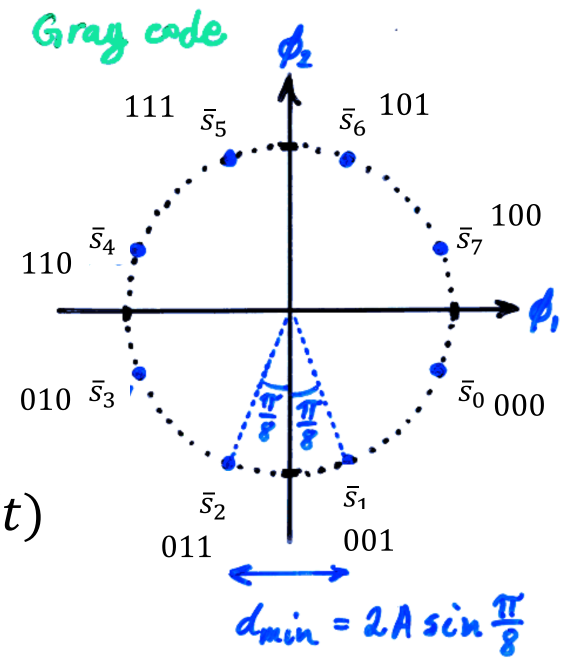
$$P_e \approx 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2Q\left(\frac{A}{\sqrt{N_0}}\right) = 2Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right) = 2Q\left(\sqrt{\frac{E_{\max}}{N_0}}\right)$$

$$P_b \approx \frac{P_e}{2} \approx Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$



M Phase-Shift Keying (M-PSK)

Example: 8-PSK



Basis:

$$\phi_0(t) = \sqrt{2/T}\cos(2\pi f_c t), \quad 0 \le t < T$$

$$\phi_1(t) = \sqrt{2/T}\sin(2\pi f_c t), \quad 0 \le t < T$$

Signals ($i = 0,1,2,3$):

$$s_i(t) = A\cos\left(\frac{(2i+1)\pi}{M}\right)\phi_0(t) - A\sin\left(\frac{(2i+1)\pi}{M}\right)\phi_1(t)$$

Energies:

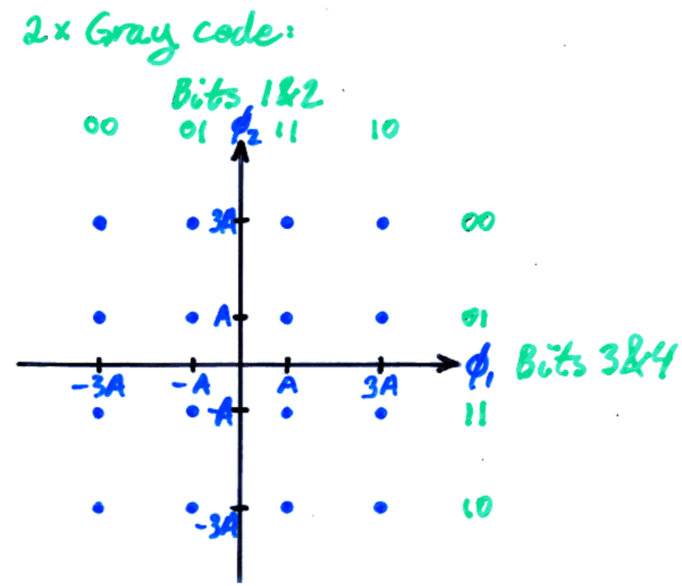
$$E_i = E_{\text{avg}} = E_{\text{max}} = A^2, \;\; E_b = A^2/k$$

Error probability (AWGN):

$$P_e \approx 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2Q\left(\frac{\sqrt{2}A}{\sqrt{N_0}}\sin\left(\frac{\pi}{M}\right)\right) = 2Q\left(\sqrt{\frac{2E_{\text{avg}}}{N_0}}\sin\left(\frac{\pi}{M}\right)\right) = 2Q\left(\sqrt{\frac{2E_{\max}}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$

$$P_b \approx \frac{P_e}{k} \approx \frac{2}{k}Q\left(\sqrt{\frac{2E_{\text{avg}}}{N_0}}\sin\left(\frac{\pi}{M}\right)\right) = \frac{2}{k}Q\left(\sqrt{\frac{2kE_b}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$

16-QAM (Quadrature Amplitude Modulation)



Basis:

$$\phi_0(t) = \sqrt{2/T}\cos(2\pi f_c t), \quad 0 \le t < T$$

$$\phi_1(t) = \sqrt{2/T}\sin(2\pi f_c t), \quad 0 \le t < T$$

Signals:

$$s_{i,j}(t) = s_i\phi_0(t) + s_j\phi_1(t),\, s_i, s_j \in \{\pm A, \pm 3A\}$$

Energies:

$$E_i = s_i^2 + s_j^2, E_{\text{avg}} = 10A^2, E_{\max} = 18A^2,\; E_b = E_{\text{avg}}/4$$

Error probability (AWGN):

$$P_e \approx \frac{4\cdot 4 + 8\cdot 3 + 4\cdot 2}{16} Q\left(\frac{2A}{\sqrt{2N_0}}\right) = 3Q\left(\sqrt{\frac{E_{\text{avg}}}{5N_0}}\right) = 3Q\left(\sqrt{\frac{E_{\max}}{9N_0}}\right)$$

$$P_b \approx \frac{P_e}{4} \approx \frac{3}{4} Q\left(\sqrt{\frac{E_{\text{avg}}}{5N_0}}\right) = \frac{3}{4} Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

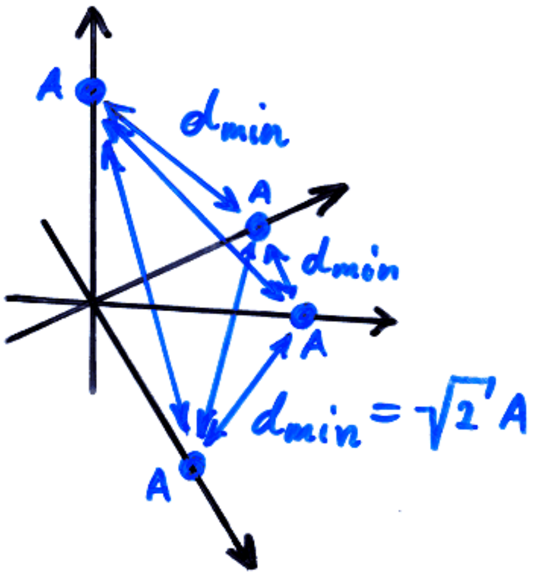
Frequency-Shift Keying (FSK)

Basis:

$$\phi_i(t) = \sqrt{2/T}\cos\left(2\pi(f_c + \frac{i}{2T})t\right), \quad 0 \le t < T$$

$$i = \{0,1,\dots,M-1\}$$

$M = 2^k$



Signals:

$$s_i(t) = A \cdot \phi_i(t)$$

Energies:

$$E_i = E_{\text{avg}} = E_{\max} = A^2,\ E_b = A^2/k$$

Error probability (AWGN):

$$P_e \approx (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right) = (2^k-1)Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right)$$

$$P_b \approx 2^{k-1}Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right) = 2^{k-1}Q\left(\sqrt{\frac{kE_b}{N_0}}\right)$$

Orthogonal Frequency Division Multiplex (OFDM)

- Principle for N dimensional signal
  - Use many 2-dimensional modulations (e.g., PSK/QAM)
  - Use $N/2$ different frequencies:

$$f_k = f_0 + \frac{k}{T}, \qquad 0 \leq k < N/2,$$

for base frequency f_0 and $2f_0T$ being an integer

- OFDM signal generation

$$s(t) = \sum_{k=0}^{\frac{N}{2}-1} \left(s_{2k}\cos(2\pi f_k t) - s_{2k+1}\sin(2\pi f_k t)\right) I_{\{0\leq t<T\}}(t)$$

for $0 \leq t < T$

OFDM continued

Alternative representation

$$s(t) = \sum_{k=0}^{\frac{N}{2}-1} \left(s_{2k}\cos(2\pi f_k t) - s_{2k+1}\sin(2\pi f_k t)\right) I_{\{0\le t<T\}}(t)$$

$$= \sum_{k=0}^{\frac{N}{2}-1} \mathrm{Re}\left\{\tilde{s}_k e^{\frac{j2\pi k}{T}t} e^{j2\pi f_0 t}\right\} I_{\{0\le t<T\}}(t) \quad \text{with } \tilde{s}_k = s_{2k} + j s_{2k+1}$$

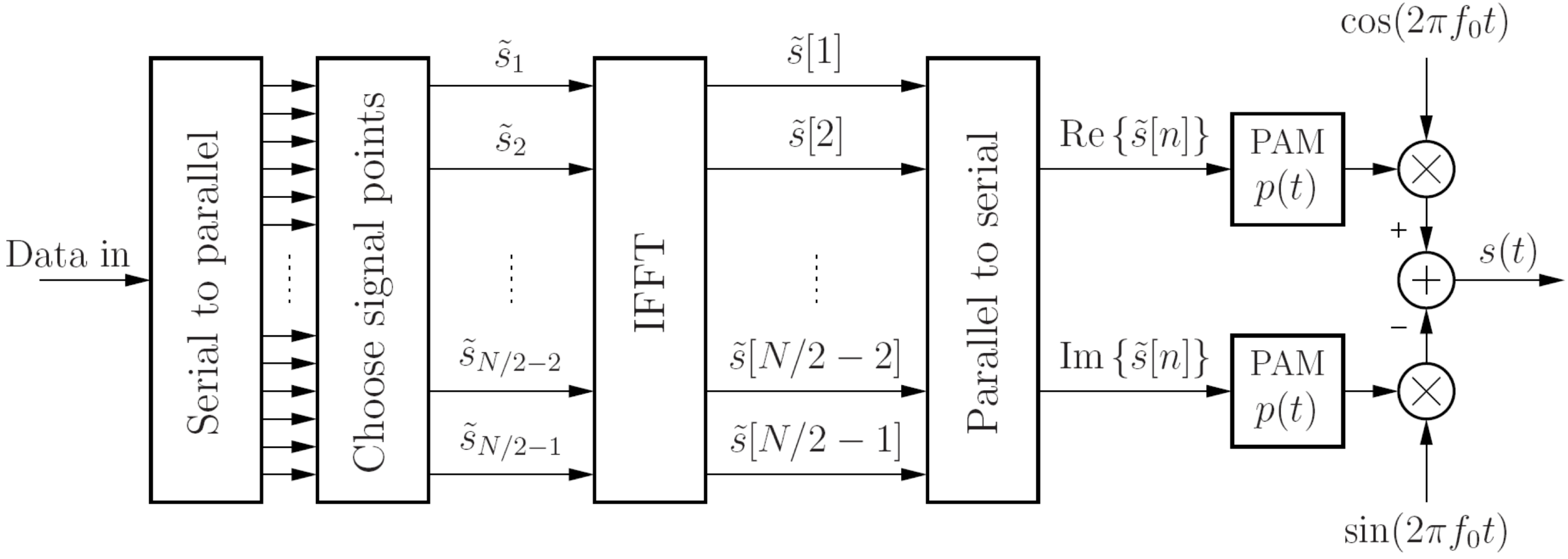
Complex baseband representation:

- $\tilde{s}(t) = \sum_{k=0}^{\frac{N}{2}-1} \tilde{s}_k e^{\frac{j2\pi k}{T}t}$ for $0 \le t < T$
- Sampled at time $t = nT/(\frac{N}{2})$: $\tilde{s}[n] = \tilde{s}\left(\frac{nT}{N/2}\right)$ for $0 \le n < N/2$
- $\tilde{s}[n] = \sum_{k=0}^{\frac{N}{2}-1} \tilde{s}_k e^{\frac{j2\pi knT}{N/2}}$ is the IDFT of $\tilde{s}_k$ for $0 \le k < N/2$

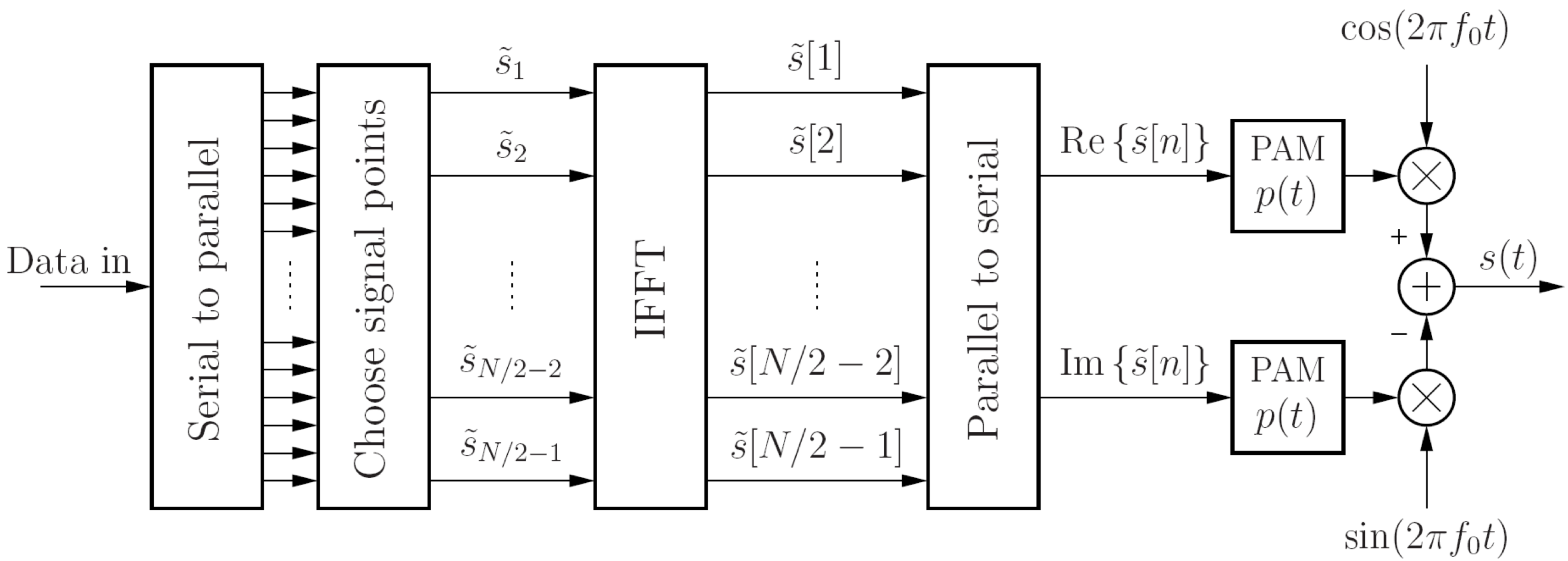
Generating an OFDM Signal

- Generate $\tilde{s}(t)$ by PAM of $\tilde{s}[n]$ (using sinc pulse function)
- Generate $s(t)$ from $\tilde{s}(t)$ *as*

$$s(t) = \mathrm{Re}\{\tilde{s}(t)e^{j2\pi f_0 t}\} = \mathrm{Re}\{\tilde{s}(t)\}\cos(2\pi f_k t) - \mathrm{Im}\{\tilde{s}(t)\}\sin(2\pi f_k t)$$



Detection of an OFDM Signal



Detector – Opposite process

- Sample
- FFT
- ML decisions for each frequency separately

Example: LTE (4G)

- Bandwidth: 10 MHz (effectively: 9 MHz)
  - OFDM with $N = 1202$
  - $N/2$ subcarriers with 2-dimensional constellations
  - Constellations: BPSK, QPSK, 16-QAM, 64-QAM
  - $9 \cdot 10^6$ symbols per second

- Data rate (no errors)
  - BPSK: $9 \cdot 10^6 = 9$ Mbit/s
  - QPSK: $2 \cdot 9 \cdot 10^6 = 18$ Mbit/s
  - 16-QAM: $4 \cdot 9 \cdot 10^6 = 36$ Mbit/s
  - 64-QAM: $6 \cdot 9 \cdot 10^6 = 54$ Mbit/s

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