

# TSDT14 Signal Theory

## Lecture 8

### Analog Modulation – Part 2

#### Repetition of Sampling and PAM (deterministically)

#### Sampling and PAM of Stochastic Processes

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## Demodulating Phase Modulation

**Sent signal:**  $x(t) = A \cdot \cos(2\pi f_c t + \phi\{m(t)\})$

**Derivative:**  $\frac{d}{dt}x(t) = -A \left(2\pi f_c + \frac{d}{dt}\phi\{m(t)\}\right) \sin(2\pi f_c t + \phi\{m(t)\})$

**Envelope detector gives us:**  $A \left(2\pi f_c + \frac{d}{dt}\phi\{m(t)\}\right)$

**BP or HP filter gives us:**  $A \frac{d}{dt}\phi\{m(t)\}$

**For PM:**  $\phi\{m(t)\} = am(t).$

**Integrate:**  $\int_{t_0}^t A \frac{d}{d\tau}\phi\{m(\tau)\} d\tau = Aa(m(t) - m(t_0))$  **Filter!**



## Demodulating Frequency Modulation

**As for PM:** Derivative, envelope detection and HP filter gives us:

$$A \frac{d}{dt}\phi\{m(t)\}$$

**For FM:**  $\phi\{m(t)\} = a \int m(t) dt.$

**Then:**  $A \frac{d}{dt}\phi\{m(t)\} = A \frac{d}{dt} a \int m(t) dt = Aa m(t)$  **Done!**



## Alternative Demodulation

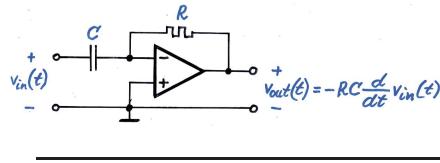


$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

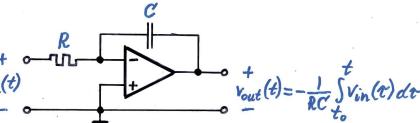


## Building Blocks

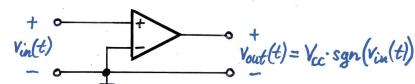
### Derivative



### Integration



### The Sign function



## Broadband Angle Modulation

### Modulation of a Gaussian process

**Assumptions:**  $\Phi_d(t)$  is an ergodic Gaussian process, mean 0.

$\Psi$  is uniform on  $[0, 2\pi]$  and indep of  $\Phi_d(t)$ .

**Signal:**  $X(t) = A \cos(2\pi f_c t + \Phi_d(t) + \Psi)$   $\Rightarrow m_X = 0$

**Phase dev.:**  $\Phi_d(t)$

**Freq. dev.:**  $F_d(t) = \frac{1}{2\pi} \cdot \frac{d}{dt} \Phi_d(t)$

**Relation:**  $R_{F_d}(f) = f^2 R_{\Phi_d}(f)$   $\Rightarrow r_{F_d}(\tau) = -\frac{1}{(2\pi)^2} \cdot \frac{d^2}{d\tau^2} r_{\Phi_d}(\tau)$

**Definitions:**  $\phi_d^2 = 2r_{\Phi_d}(0)$   $f_d^2 = 2r_{F_d}(0)$ ,

## ACF of Angle Modulation 1(3)

$$\begin{aligned} r_X(t + \tau, t) &= E \left\{ A \cos(2\pi f_c(t + \tau) + \Phi_d(t + \tau) + \Psi) \cdot A \cos(2\pi f_c t + \Phi_d(t) + \Psi) \right\} \\ &= \frac{A^2}{2} E \left\{ \cos(2\pi f_c(2t + \tau) + \Phi_d(t + \tau) + \Phi_d(t) + 2\Psi) \right\} \\ &\quad + \frac{A^2}{2} E \left\{ \cos(2\pi f_c\tau + \Phi_d(t + \tau) - \Phi_d(t)) \right\}, \\ &= \frac{A^2}{2} E \left\{ \cos(2\pi f_c\tau + \Phi_d(t + \tau) - \Phi_d(t)) \right\} \\ &= \frac{A^2}{2} E \left\{ \operatorname{Re} \left\{ e^{j[2\pi f_c\tau + \Phi_d(t + \tau) - \Phi_d(t)]} \right\} \right\} \\ &= \frac{A^2}{2} \operatorname{Re} \left\{ E \left\{ e^{j[\Phi_d(t + \tau) - \Phi_d(t)]} \right\} e^{j2\pi f_c\tau} \right\}, \end{aligned}$$

## ACF of Angle Modulation 2(3)

We had:  $r_X(t + \tau, t) = \frac{A^2}{2} \operatorname{Re} \left\{ E \left\{ e^{j[\Phi_d(t + \tau) - \Phi_d(t)]} \right\} e^{j2\pi f_c\tau} \right\}$

Temporary Gaussian variable:  $Y = \Phi_d(t + \tau) - \Phi_d(t)$  (mean 0)

Then:  $E \{ e^{jY} \} = E \{ \cos(Y) \} + jE \{ \sin(Y) \} = E \{ \cos(Y) \}$ ,

Important observation: This is real valued.

Rewrite:  $r_X(t + \tau, t) = \frac{A^2}{2} E \{ e^{jY} \} \operatorname{Re} \{ e^{j2\pi f_c\tau} \} = \frac{A^2}{2} E \{ e^{jY} \} \cos(2\pi f_c\tau)$

We have, with  $f = -1/2\pi$ , using the identity  $\mathcal{F}_x \{ e^{-\pi(\tilde{x}/T)^2} \} = T e^{-\pi(fT)^2}$ :

$$\begin{aligned} E \{ e^{jY} \} &= E \{ e^{-j2\pi f Y} \} = \mathcal{F}_y \{ f_Y(y) \} = \mathcal{F}_y \left\{ \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-y^2/2\sigma_Y^2} \right\} \\ &= e^{-(2\pi f \sigma_Y)^2/2} = e^{-\sigma_Y^2/2}. \end{aligned}$$

## ACF of Angle Modulation 3(3)

We had:  $r_X(t + \tau, t) = \frac{A^2}{2} e^{-\sigma_Y^2/2} \cos(2\pi f_c \tau)$

Recall:  $Y = \Phi_d(t + \tau) - \Phi_d(t)$  (mean 0)

We have:  $\sigma_Y^2 = E \left\{ (\Phi_d(t + \tau) - \Phi_d(t))^2 \right\} = 2r_{\Phi_d}(0) - 2r_{\Phi_d}(\tau)$ .

Mean and ACF independent of time  $t$ . Wide sense stationary!

$$m_X = 0 \quad r_X(\tau) = \frac{A^2}{2} \cdot r_{\tilde{X}}(\tau) \cdot \cos(2\pi f_c \tau) \quad r_{\tilde{X}}(\tau) = e^{r_{\Phi_d}(\tau) - r_{\Phi_d}(0)}$$

The introduced process  $\tilde{X}(t)$  is a normalized baseband representation of  $X(t)$ .

## PSD of Angle Modulation

We had:  $r_X(\tau) = \frac{A^2}{2} \cdot r_{\tilde{X}}(\tau) \cdot \cos(2\pi f_c \tau) \quad r_{\tilde{X}}(\tau) = e^{r_{\Phi_d}(\tau) - r_{\Phi_d}(0)}$

Recall:  $\phi_d^2 = 2r_{\Phi_d}(0) \quad f_d^2 = 2r_{F_d}(0)$ ,

This holds if  $r_{\Phi_d}(\tau)$  has no periodic components and  $\Phi_d(t)$  has mean 0:

$$r_{\tilde{X}}(\tau) \approx m_{\tilde{X}}^2 + (1 - m_{\tilde{X}}^2) \cdot e^{-2\pi^2 r_{F_d}(0)\tau^2} = m_{\tilde{X}}^2 + (1 - m_{\tilde{X}}^2) \cdot e^{-(\pi f_d \tau)^2}$$

$$m_{\tilde{X}}^2 = e^{-r_{\Phi_d}(0)} = e^{-\phi_d^2/2}.$$

Spectrum:  $R_{\tilde{X}}(f) \approx m_{\tilde{X}}^2 \delta(f) + \frac{1 - m_{\tilde{X}}^2}{\sqrt{\pi} \cdot f_d} \cdot e^{-(f/f_d)^2}$ .

Carrier      Sidebands

The only relation to  $R_{\Phi_d}(f)$  is through  $r_{F_d}(0)$  and  $r_{\Phi_d}(0)$ .

## Bandwidth of Angle Modulation

Define bandwidth  $B_X$  as the width of the frequency band, symmetrically around the carrier frequency that contains 90% of the power.

$$\int_{-B_X/2}^{B_X/2} R_{\tilde{X}}(f) df = 0.9.$$

Broadband  $\Rightarrow m_{\tilde{X}}^2$  very small. Ignore!

$$\int_{-B_X/2}^{B_X/2} \frac{1}{\sqrt{\pi} \cdot f_d} \cdot e^{-(f/f_d)^2} df \approx 0.9.$$

We get:

$$\int_{B_X/2}^{\infty} \frac{1}{\sqrt{\pi} \cdot f_d} \cdot e^{-(f/f_d)^2} df = Q\left(\frac{B_X/2}{f_d/\sqrt{2}}\right) \approx 0.05. \Rightarrow B_X \approx 2.33 f_d.$$

Compare Carson's rule: Bandwidth is slightly more than  $2 \cdot f_{d,\max}$ .

## Special Cases: Broadband PM and FM

Phase modulation:

$$\Phi_d(t) = a M(t),$$

$$r_{\Phi_d}(\tau) = a^2 r_M(\tau) = \frac{\phi_d^2}{2} \cdot \frac{r_M(\tau)}{r_M(0)},$$

$$R_{\Phi_d}(f) = a^2 R_M(f) = \frac{\phi_d^2}{2} \cdot \frac{R_M(f)}{r_M(0)}.$$

Frequency modulation

$$\Phi_d(t) = a \int M(t) dt,$$

$$F_d(t) = \frac{1}{2\pi} \cdot \frac{d}{dt} \Phi(t) = \frac{a}{2\pi} M(t),$$

$$r_{F_d}(\tau) = \left(\frac{a}{2\pi}\right)^2 r_M(\tau) = \frac{f_d^2}{2} \cdot \frac{r_M(\tau)}{r_M(0)},$$

$$R_{F_d}(f) = \left(\frac{a}{2\pi}\right)^2 R_M(f) = \frac{f_d^2}{2} \cdot \frac{R_M(f)}{r_M(0)}.$$

## Narrowband Angle Modulation

We have:

$$\begin{aligned} X(t) &= A \cos(2\pi f_c t + \Phi_d(t) + \Psi) \\ &= A \cos(\Phi_d(t)) \cos(2\pi f_c t + \Psi) - A \sin(\Phi_d(t)) \sin(2\pi f_c t + \Psi) \end{aligned}$$

Narrowband modulation, i.e.  $\Phi_d(t)$  is small:

$$X(t) \approx A \cos(2\pi f_c t + \Psi) - A \Phi_d(t) \sin(2\pi f_c t + \Psi).$$

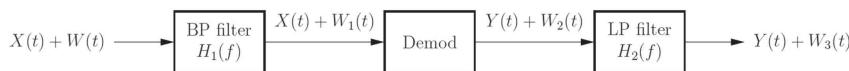
Spectrum:

$$R_X(f) = \frac{A^2}{4} (\delta(f + f_c) + \delta(f - f_c)) + \frac{A^2}{4} (R_{\Phi_d}(f + f_c) + R_{\Phi_d}(f - f_c))$$

$\uparrow$  Carrier                     $\uparrow$  Sidebands

We get AM in the so called quadrature component.

## AWGN Impact on Angle Modulation



Noise PSD:  $R_W(f) = R_0$

Filter:  $H_1(f) = \begin{cases} 1, & |f| - f_c < B_X/2, \\ 0, & \text{elsewhere.} \end{cases}$

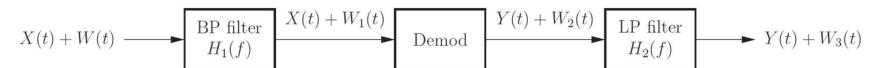
Filtered noise PSD:  $R_{W_1}(f) = |H_1(f)|^2 R_W(f) = \begin{cases} R_0, & |f| - f_c < B_X/2, \\ 0, & \text{elsewhere,} \end{cases}$

Filtered noise power:  $P_{W_1} = \int_{-\infty}^{\infty} R_{W_1}(f) df = 2 \int_{f_c - B_X/2}^{f_c + B_X/2} R_0 df = 2B_X R_0.$

Signal power:  $P_X = \frac{A^2}{2}$

Input SNR:  $\frac{P_X}{P_{W_1}} = \frac{A^2/2}{2B_X R_0} = \frac{A^2}{4B_X R_0}$

## AWGN Impact on Phase Modulation



Output signal:  $Aa m(t)$

Output signal power:  $P_Y = A^2 a^2 P_M$

Can be shown:  $R_{W_2}(f) = \begin{cases} 2R_0, & |f| < B_X/2, \\ 0, & \text{elsewhere,} \end{cases}$

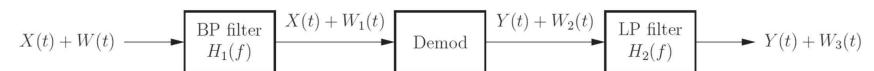
Filter:  $H_2(f) = \begin{cases} 1, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$

Output noise:  $R_{W_3}(f) = |H_2(f)|^2 R_{W_2}(f) = \begin{cases} 2R_0, & |f| < B, \\ 0, & \text{elsewhere,} \end{cases}$

Noise power:  $P_{W_3} = \int_{-\infty}^{\infty} R_{W_3}(f) df = \int_{-B}^{B} 2R_0 df = 4BR_0.$

Output SNR:  $\frac{P_Y}{P_{W_3}} = \frac{A^2 a^2 P_M}{4BR_0}$

## AWGN Impact on Frequency Modulation



Output signal:  $Aa m(t)$

Output signal power:  $P_Y = A^2 a^2 P_M$

Can be shown:  $R_{W_2}(f) = \begin{cases} 2f^2 R_0, & |f| < B_X/2, \\ 0, & \text{elsewhere,} \end{cases}$

Filter:  $H_2(f) = \begin{cases} 1, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$

Output noise:  $R_{W_3}(f) = \begin{cases} 2f^2 R_0, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$

Noise power:  $P_{W_3} = \frac{4B^3 R_0}{3}.$

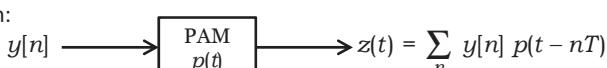
Output SNR:  $\frac{P_Y}{P_{W_3}} = \frac{3A^2 a^2 P_M}{4B^3 R_0},$

## Linear Mappings

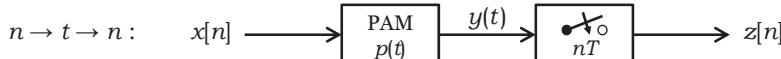
Sampling:



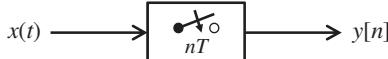
Pulse-Amplitude Modulation:  
(PAM)



Reconstruction:



## Sampling – Deterministically (from S&S)



Sampling frequency:  $f_s = 1/T$

Time domain:  $y[n] = x(nT)$

Frequency domain:  $Y[\theta] = \frac{1}{T} \sum_m X\left(\frac{\theta - m}{T}\right) = f_s \sum_m X((\theta - m)f_s)$

If  $X(f) = 0$  for  $|f| \geq f_s/2$ :

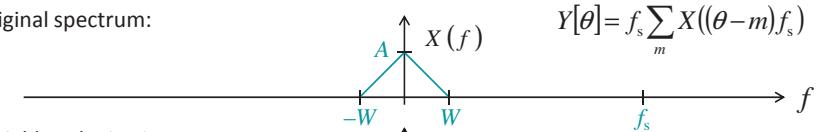
$$Y[\theta] = f_s X(\theta f_s) \quad \text{for } |\theta| < 1/2$$

$$Y[\theta] = Y[\theta + k] \quad \text{for } k \text{ integer}$$

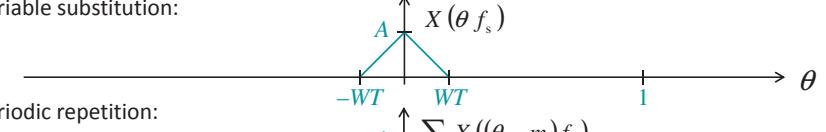


## Sampling – Frequency Domain

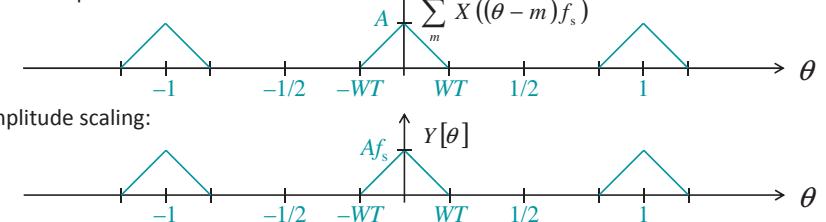
Original spectrum:



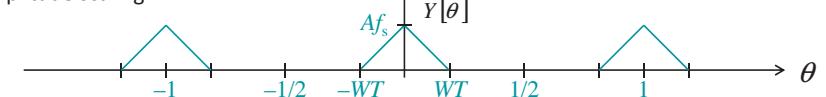
Variable substitution:



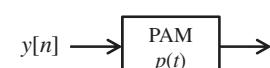
Periodic repetition:



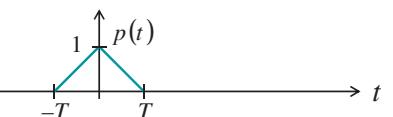
Amplitude scaling:



## PAM – Deterministically (from S&S)

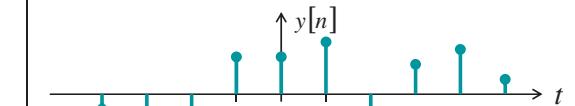


Example:



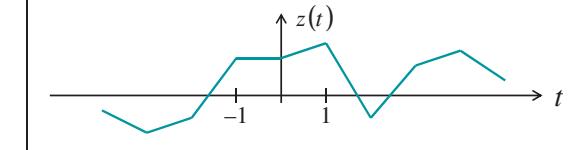
Time domain:

$$z(t) = \sum_n y[n] p(t - nT)$$

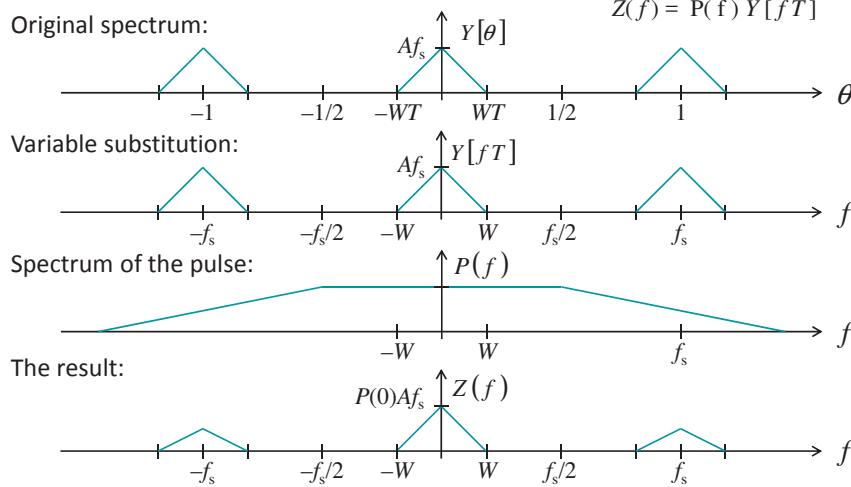


Frequency domain:

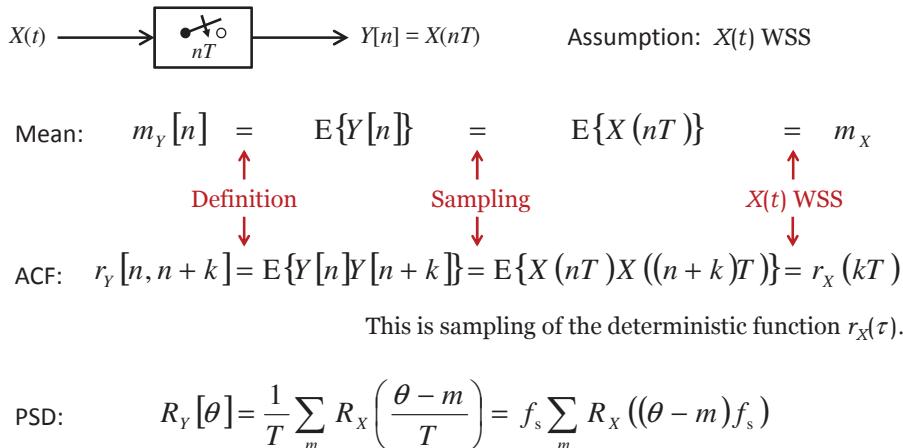
$$Z(f) = P(f) Y[fT]$$



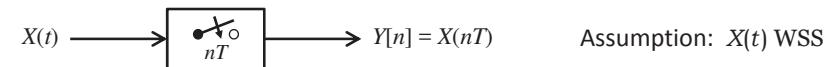
## PAM – Frequency Domain



## Sampling of a Stochastic Process 1(2)



## Sampling of a Stochastic Process 2(2)



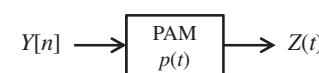
$$R_Y[\theta] = \frac{1}{T} \sum_m R_x\left(\frac{\theta - m}{T}\right) = f_s \sum_m R_x((\theta - m)f_s)$$

If  $R_X(f) = 0$  for  $|f| \geq f_s/2$ :

$$R_Y[\theta] = f_s R_X(\theta f_s) \text{ for } |\theta| < 1/2$$

$$R_Y[\theta] = R_Y[\theta + k] \text{ for } k \text{ integer}$$

## PAM of a Stochastic Process 1(3)



Time domain (alternative approach):  $\Psi$  unif.  $[0, T]$

$$Z(t) = \sum_n Y[n] p(t - nT - \Psi) \quad \Psi \& Y[n] \text{ indep.}$$

Mean:

$$m_Z(t) = E\{Z(t)\} = E\left\{\sum_n Y[n] p(t - nT - \Psi)\right\}$$

$$= \sum_n E\{Y[n]\} E\{p(t - nT - \Psi)\} \quad (\text{indep.})$$

$$= \sum_n m_Y \int_{-\infty}^T p(t - nT - \psi) f_\Psi(\psi) d\psi$$

$$= m_Y \sum_n \int_0^\infty p(t - nT - \psi) \frac{1}{T} d\psi = \int_0^\infty p(t - nT - \psi) \frac{1}{T} d\psi / \int_0^\infty d\psi = \int_0^\infty p(t - nT - \psi) \frac{1}{T} d\psi$$

$$= m_Y \frac{1}{T} \sum_n \int_{t-nT-T}^{t-nT} p(\phi) d\phi = m_Y \frac{1}{T} \int_{-\infty}^\infty p(\phi) d\phi$$

$$= \frac{1}{T} P(0) m_Y$$

Independent of  $t$ .

## PAM of a Stochastic Process 2(3)

ACF:

$$\begin{aligned}
 r_z(t + \tau, t) &= E\{Z(t + \tau)Z(t)\} = E\left\{\sum_n Y[n]p(t + \tau - nT - \Psi) \sum_m Y[m]p(t - mT - \Psi)\right\} \\
 &= \sum_n \sum_m E\{Y[n]Y[m]\}E\{p(t + \tau - nT - \Psi)p(t - mT - \Psi)\} \\
 &= \sum_n \sum_m r_Y[n - m] \int_0^T p(t + \tau - nT - \psi)p(t - mT - \psi) \frac{1}{T} d\psi = \left/ \begin{array}{l} k = n - m \\ \psi = \psi + mT - t \end{array} \right. \\
 &= \frac{1}{T} \sum_k r_Y[k] \sum_m \int_0^T p(t + \tau - (k + m)T - \psi)p(t - mT - \psi) d\psi = \left/ \begin{array}{l} \phi = \psi + mT - t \\ d\phi = d\psi \end{array} \right. \\
 &= \frac{1}{T} \sum_k r_Y[k] \sum_m \int_{mT-t}^{T+mT-t} p(\tau - kT - \phi)p(-\phi) d\phi = \left/ \begin{array}{l} \text{define } \tilde{p}(t) = p(-t) \\ \psi = \phi - mT \end{array} \right. \\
 &= \frac{1}{T} \sum_k r_Y[k] \int_{-\infty}^{\infty} p(\tau - kT - \phi)\tilde{p}(\phi) d\phi = \frac{1}{T} \sum_k r_Y[k](p * \tilde{p})(\tau - kT)
 \end{aligned}$$

Independent of  $t$ .



## PAM of a Stochastic Process 3(3)

**Mean:**  $m_z = \frac{1}{T} P(0)m_y$

Independent of  $t$ .  
Thus WSS

**ACF:**  $r_z(\tau) = \frac{1}{T} \sum_k r_Y[k](p * \tilde{p})(\tau - kT)$

**PSD:**  $R_z(f) = \mathcal{F}\left\{\frac{1}{T} \sum_k r_Y[k](p * \tilde{p})(\tau - kT)\right\}$

**Notice:** The sum is PAM of  $r_Y[k]$  using pulse  $(p * \tilde{p})(\tau)$ .

$$\mathcal{F}\{(p * \tilde{p})(\tau)\} = |P(f)|^2$$

**Recall deterministic PAM:**

$$R_z(f) = \frac{1}{T} |P(f)|^2 R_Y[fT]$$



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