TSKS01 Digital Communication Lecture 8

Error Control Codes

Emil Björnson

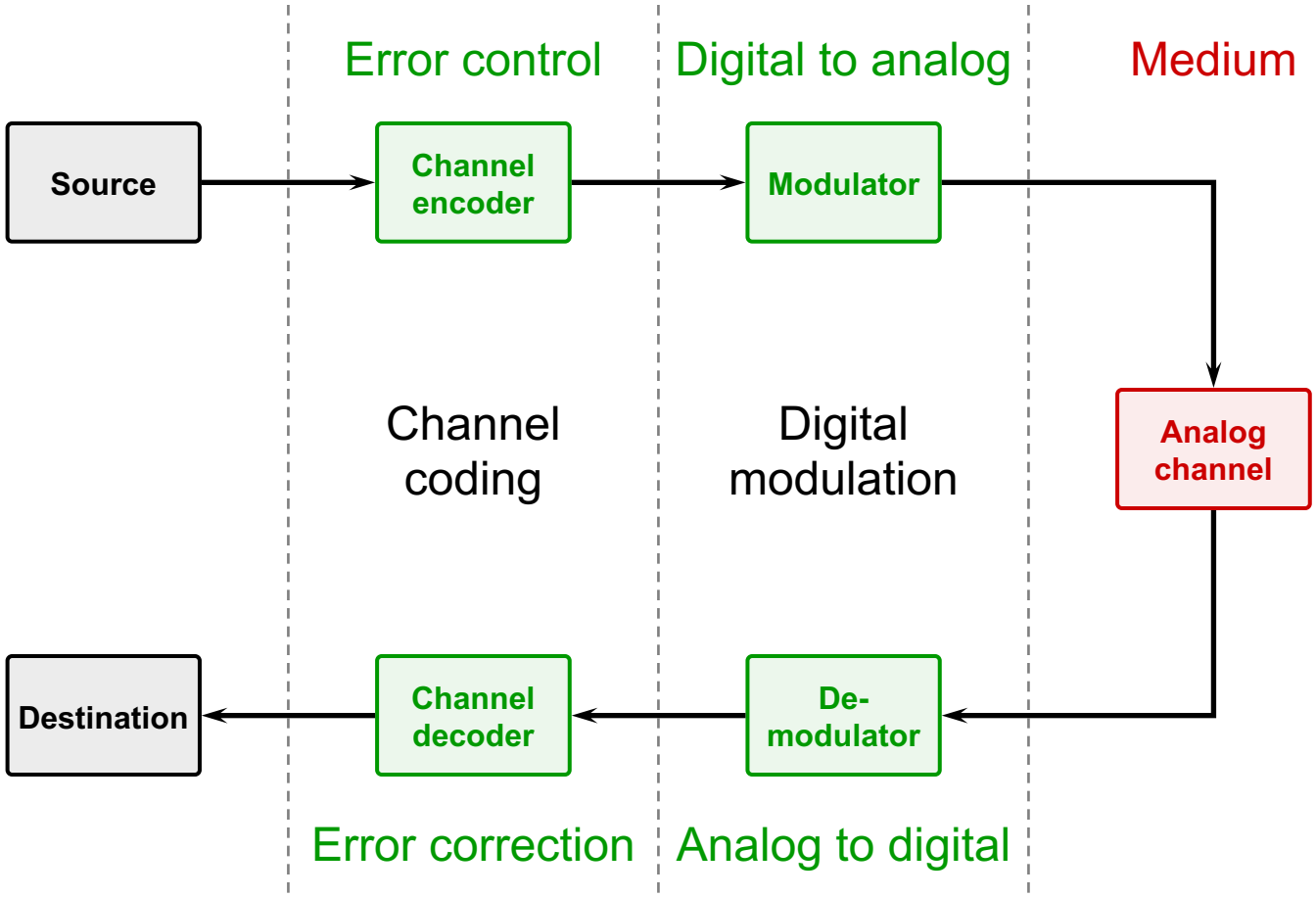
Department of Electrical Engineering (ISY)

Division of Communication Systems

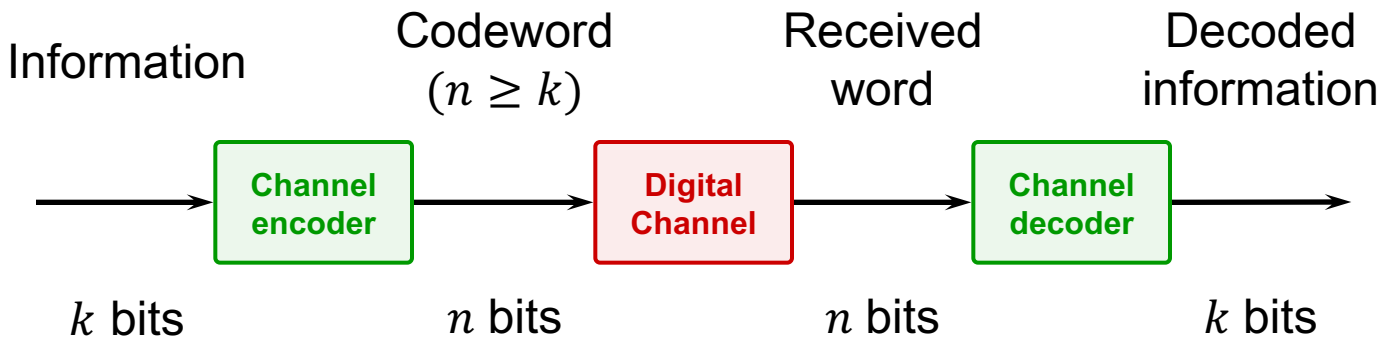
COMMUNICATION SYSTEMS

LINKÖPING UNIVERSITY

A One-way Digital Communication System



Block Codes – Basic Idea



Definition: A *parity bit* of a string indicates if the number of ones is even or odd

Calculate r parity bits from k information bits.

Send $n = k + r$ codeword bits.

Received: n possibly corrupted bits.

More general:
Map k information bits on $n = k + r$ codeword bits.

Decode to the most likely sent codeword given the received bits.

If the codewords are well chosen, then we will be able to correct errors.

Block Code – Set of Binary Vectors

Code: $\mathcal{C} = \{\bar{c}_i \in \{0,1\}^n\}_{i=1}^{2^k}$

Codeword: $\bar{c}_i = (c_{i,1}, \ldots, c_{i,n})$

$\in \{0,1\}$

Size: $M = 2^k$ messages

Rate: $R = \frac{k}{n}$

Example code:

Information	Codeword
00	10101010
01	11010000
10	01100111
11	00011101
$k = 2$	$n = 8$

Decoding principles:

Assume that all errors are equally serious. $\Rightarrow$

Choose the most likely codeword given the received vector.

Maximum Likelihood Decoding

Stochastic variables: Sent codeword: $\overline{C}$

Received word: $\overline{X}$

General decoding rule:

Decoding rule 1: Set $\hat{\bar{c}} = \bar{c}_i$ if $\Pr\left\{\overline{C} = \bar{c}_k \mid \overline{X} = \bar{x}\right\}$ is maximized for $k = i$.

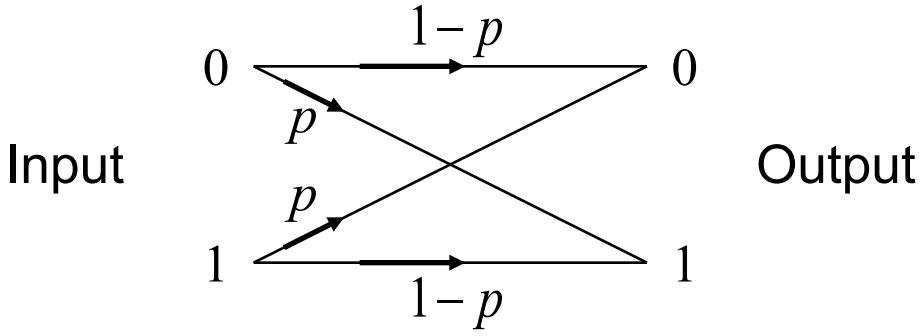
Bayes rule $\Rightarrow$

Decoding rule 2: Set $\hat{\bar{c}} = \bar{c}_i$ if $\Pr\left\{\overline{C} = \bar{c}_k\right\}\Pr\left\{\overline{X} = \bar{x} \mid \overline{C} = \bar{c}_k\right\}$ is max for $k = i$.

ML decoding $\left(\Pr\left\{\overline{C} = \bar{c}_k\right\} = 1/2^k\right)$:

Decoding rule 3: Set $\hat{\bar{c}} = \bar{c}_i$ if $\Pr\left\{\overline{X} = \bar{x} \mid \overline{C} = \bar{c}_k\right\}$ is maximized for $k = i$.

The Binary Symmetric Channel (BSC)



Consecutive uses of the channel are independent

Binary modulation schemes gives a BSC!

Hamming distance: $d_{\mathrm{H}}(\bar{a},\bar{b})$ # positions where $\bar{a}$ and $\bar{b}$ differ.

Properties: $d_{\mathrm{H}}(\bar{a},\bar{a})=0 \quad d_{\mathrm{H}}(\bar{a},\bar{b})\geq 0 \quad d_{\mathrm{H}}(\bar{a},\bar{c})\leq d_{\mathrm{H}}(\bar{a},\bar{b})+d_{\mathrm{H}}(\bar{b},\bar{c})$

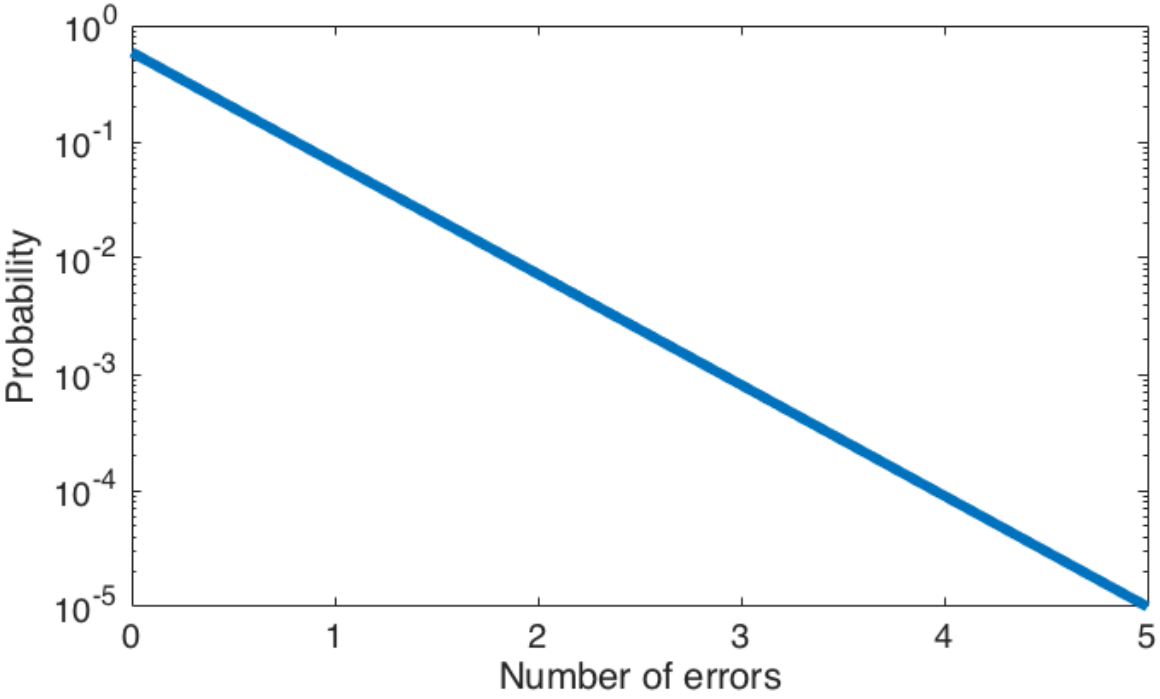
We get: $\Pr\{\bar{X}=\bar{x}\,|\,\bar{C}=\bar{c}_k\}=p^{d_{\mathrm{H}}(\bar{x},\bar{c}_k)}(1-p)^{n-d_{\mathrm{H}}(\bar{x},\bar{c}_k)}$

ML decoding for BSC with error probability p (assuming $0<p<0.5$):

Decoding rule 4: Set $\hat{\bar{c}}=\bar{c}_i$ if $d_{\mathrm{H}}(\bar{x},\bar{c}_k)$ is minimized for $k=i$.

Error Probability with BSC

Probability of j errors: $\mathrm{p}^{j}(1-p)^{n-j}$



Here: $p = 0.1, n = 5$

Few errors is much more probability than many errors

Example of Decoding

Information	Codeword
00	10101010
01	11010000
10	01100111
11	00011101
$k = 2$	$n = 8$

$d_H(10101010,11010000) = 5$
$d_H(10101010,01100111) = 5$
$d_H(10101010,00011101) = 6$
$d_H(11010000,01100111) = 6$
$d_H(11010000,00011101) = 5$
$d_H(01100111,00011101) = 5$

Minimum distance $d = 5$

Decoding: Choose closest codeword.

(10111010) is decoded to (10101010)
(11110111) is decoded to (01100111)

(00111000) is on distance 3 from both (10101010) & (00011101)

Error correction capability

$$t = \left\lfloor \frac{d-1}{2} \right\rfloor$$ Here $t = 2$

Error detection capability

$v = d - 1$ Here $v = 4$

The Binary Field, $\mathbb{F}_2$

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Integer arithmetic reduced modulo 2

Associative law: $(a+b)+c = a+(b+c) \quad \forall\, a, b, c \in \mathbb{F}_2$

$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall\, a, b, c \in \mathbb{F}_2$

Commutative law: $a+b = b+a \quad \forall\, a, b \in \mathbb{F}_2$

$a \cdot b = b \cdot a \quad \forall\, a, b \in \mathbb{F}_2$

Unit elements: $\exists\, 0 \in \mathbb{F}_2: \quad a+0 = a \quad \forall\, a \in \mathbb{F}_2$

$\exists\, 1 \in \mathbb{F}_2: \quad a \cdot 1 = a \quad \forall\, a \in \mathbb{F}_2$

Inverses: $\exists\, -a \in \mathbb{F}_2: \quad a+(-a) = 0 \quad \forall\, a \in \mathbb{F}_2$

$\exists\, a^{-1} \in \mathbb{F}_2: \quad a \cdot a^{-1} = 1 \quad \forall\, a \in \mathbb{F}_2, a \neq 0$

Distributive law: $a \cdot (b+c) = a \cdot b + a \cdot c \quad \forall\, a, b, c \in \mathbb{F}_2$

Binary Linear Codes

Definition:
A binary linear code is a linear subspace of the full vector space $\mathbb{F}_2^n$.

Equivalent definition:
The binary code $\mathcal{C}$ is called linear if $\bar{c}_1 + \bar{c}_2 \in \mathcal{C}$ holds for all $\bar{c}_1, \bar{c}_2 \in \mathcal{C}$.

Generator matrix G:

$$\mathcal{C} = \{\bar{m}G \ \ \forall \bar{m} \in \mathbb{F}_2^k\}$$

Generator matrix, $k \times n$

Information vector, k bits

Example:

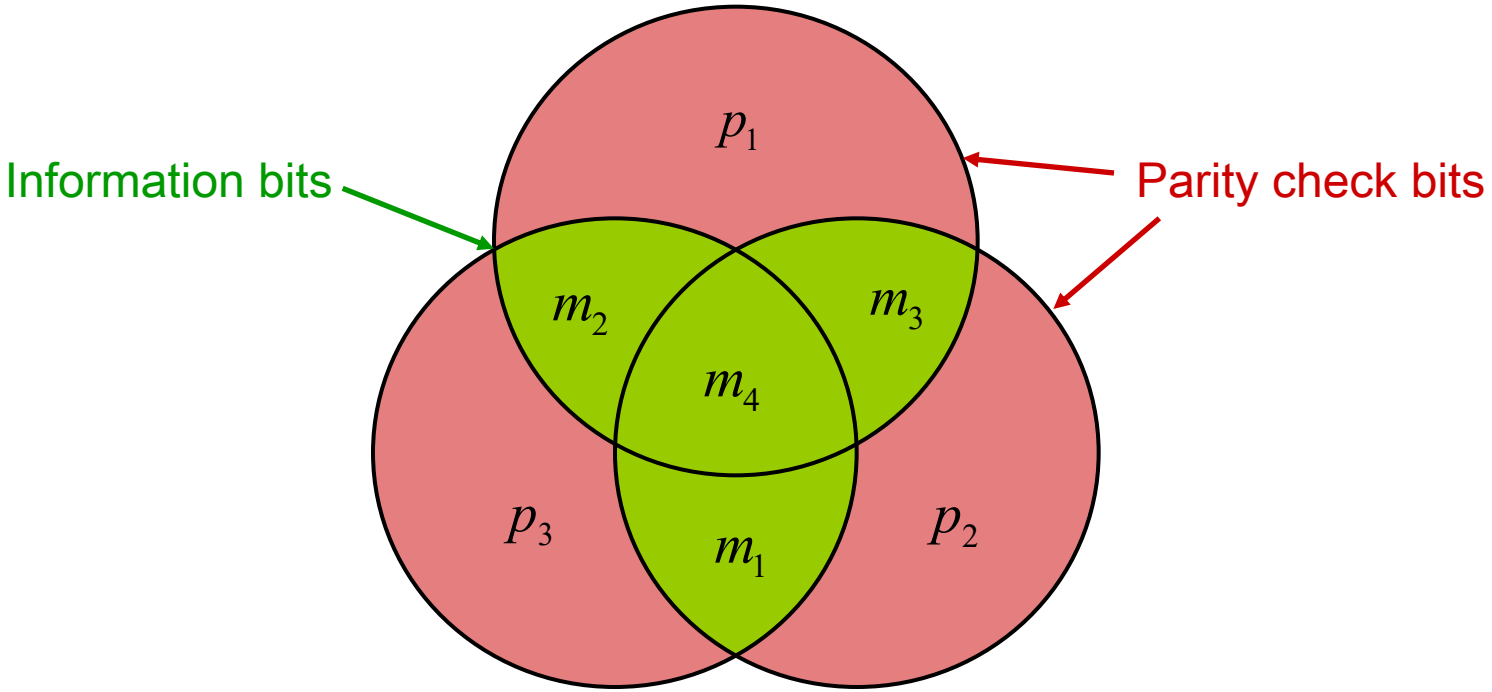
$$G = \begin{pmatrix} 1100 \\ 0111 \end{pmatrix}$$

$\bar{m}$	$\bar{c} = \bar{m}G$
00	0000
01	0111
10	1100
11	1011

We can use everything from linear algebra!

The [7,4] Hamming Code

$$k = 4, n = 7$$



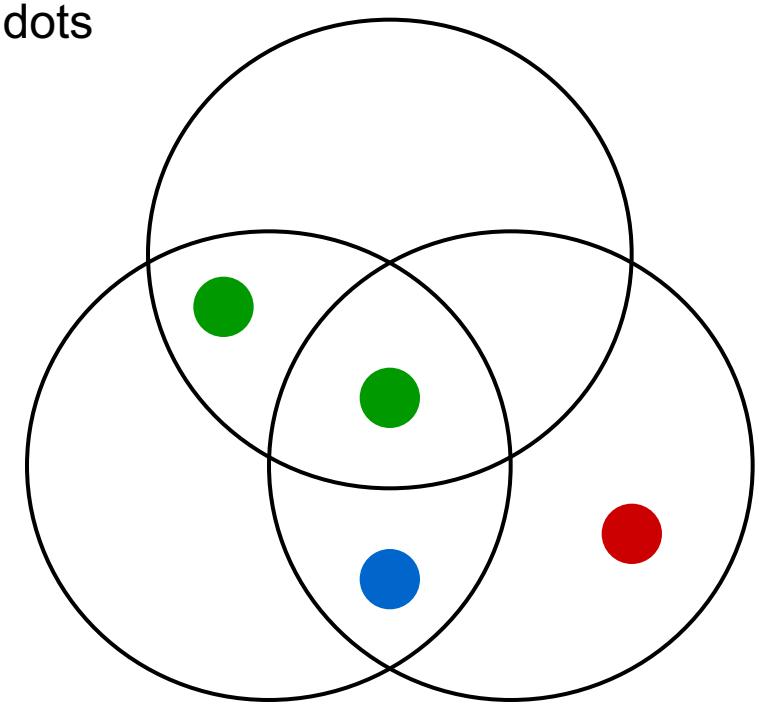
Even parity in each circle. $\Rightarrow$

$$\begin{cases} p_1 + m_2 + m_3 + m_4 = 0 \pmod 2 \\ p_2 + m_1 + m_3 + m_4 = 0 \pmod 2 \\ p_3 + m_1 + m_2 + m_4 = 0 \pmod 2 \end{cases}$$

The [7,4] Hamming Code – Example

Ones are marked with dots

Information

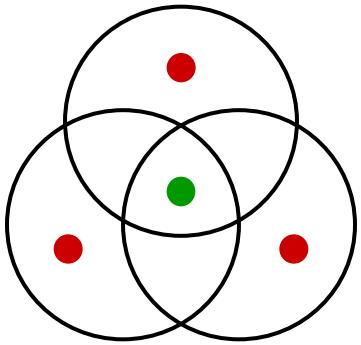


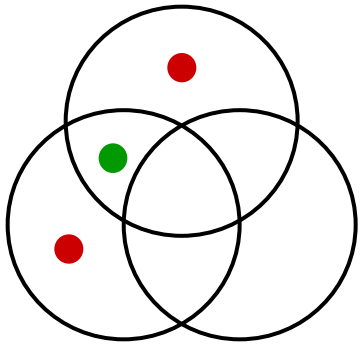
Even parity in each circle

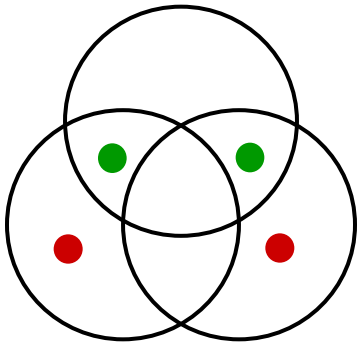
An error!

Result: Even parity in the upper circle, odd parity in the two lower circles.
Only one position can explain that – the actual error.

The [7,4] Hamming Code – More Examples

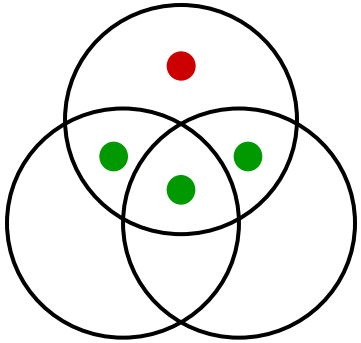


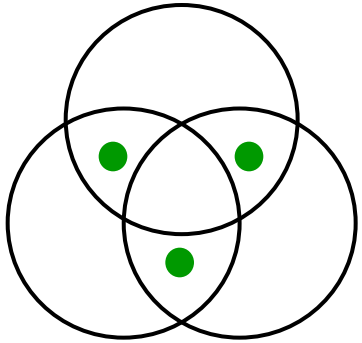


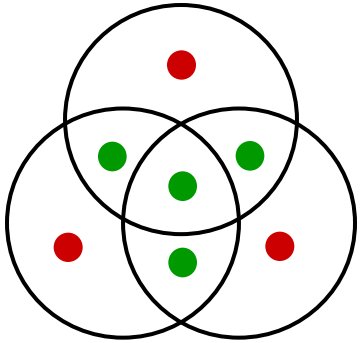


Information

Even parity in each circle.







[7,4] Hamming Code – Generator Matrix

$$\begin{cases} p_1 + m_2 + m_3 + m_4 = 0 \pmod 2 \\ p_2 + m_1 + m_3 + m_4 = 0 \pmod 2 \\ p_3 + m_1 + m_2 + m_4 = 0 \pmod 2 \end{cases} \Rightarrow \begin{cases} p_1 = m_2 + m_3 + m_4 \pmod 2 \\ p_2 = m_1 + m_3 + m_4 \pmod 2 \\ p_3 = m_1 + m_2 + m_4 \pmod 2 \end{cases}$$

Codeword:

$$\bar{c} = (m_1, m_2, m_3, m_4, p_1, p_2, p_3) = (m_1, m_2, m_3, m_4) \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Generator matrix, G, on systematic form

$k \times k$ identity matrix

$$G = (I_k, P) \quad \text{with} \quad P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Code:

$$\mathcal{C} = \{\bar{c} = \bar{m}G \ \ \forall \bar{m} \in \mathbb{F}_2^4\}$$

Dimension:
k = # rows in G. Here 4.

Length:
n = # columns in G. Here 7.

Nullspaces and Parity Check Matrices

A vector space (code) expressed in a basis:

$$\mathcal{C} = \{\bar{m}G \;\; \forall \bar{m} \in \mathbb{F}_2^k\}$$

Generator matrix ($k \times n$), linearly independent rows.

A vector space (code) expressed as the nullspace of a matrix:

$$\mathcal{C} = \{\bar{c} \in \mathbb{F}_2^n : H\bar{c}^T = \bar{0}\}$$

Parity check matrix ($(n - k) \times n$), linearly independent rows.

Property: $HG^T = 0$

[7,4] Hamming Code – Parity Check Matrix

$$\begin{cases} m_2 + m_3 + m_4 + p_1 = 0 \\ m_1 + m_3 + m_4 + p_2 = 0 \\ m_1 + m_2 + m_4 + p_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{cccc:ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Parity check matrix, H, on systematic form

Compare to the generator matrix:

$$G = \left(\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) = (I_4, P)$$

$$H = (P^{\mathrm{T}}, I_3)$$

In general (systematic form):

$$\begin{cases} G = (I_k, P) \\ H = (P^{\mathrm{T}}, I_{n-k}) \end{cases} \quad \text{or} \quad \begin{cases} G = (P, I_k) \\ H = (I_{n-k}, P^{\mathrm{T}}) \end{cases}$$

Codes may have non-systematic generator and/or parity check matrices

Weights and Distances

Hamming weight: $w_{\mathrm{H}}(\bar{a})$ # positions where $\bar{a}$ is 1 (non-zero).

Hamming distance: $d_{\mathrm{H}}(\bar{a},\bar{b})$ # positions where $\bar{a}$ and $\bar{b}$ differ.

Relation: $d_{\mathrm{H}}(\bar{a},\bar{b}) = w_{\mathrm{H}}(\bar{a}+\bar{b})$

Linear code

Minimum distance: $$d = \min_{i\neq j} d_{\mathrm{H}}(\bar{c}_i,\bar{c}_j) = \min_{i\neq j} w_{\mathrm{H}}(\bar{c}_i+\bar{c}_j) = \min_{\substack{\bar{c}\in\mathscr{C}\\ \bar{c}\neq\bar{0}}} w_{\mathrm{H}}(\bar{c})$$

Also: d is the smallest number of linearly dependent columns in H (since $H\bar{c}^T = \bar{0}$)

Example Hamming [7,4]:

$$H = \left(\begin{array}{ccc:ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

No column $= \bar{0} \Rightarrow d > 1$

No two columns equal

$\bar{h}_1 + \bar{h}_6 + \bar{h}_7 = \bar{0}$

$\Rightarrow d > 2 \Rightarrow d = 3$

Error Correction and Detection Capability

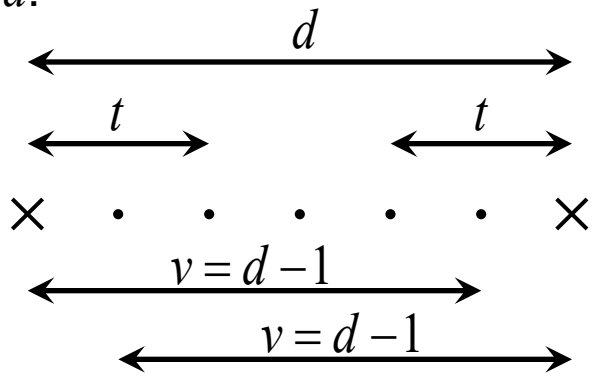
Error correction capability: $t = \left\lfloor \frac{d-1}{2} \right\rfloor$

The code can correct every w-bit error if $w \leq t$.

Error detection capability: $v = d - 1$

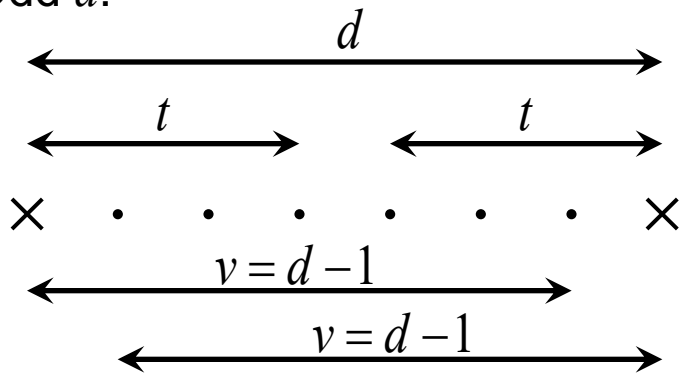
The code can detect every w-bit error if $w \leq v$.

Even d:



$$t = \frac{d-2}{2}$$

Odd d:



$$t = \frac{d-1}{2}$$

Overview – Binary Linear Codes $[n, k, d]$

A vector space expressed in a basis

$$\mathcal{C} = \{\bar{m}G \;\; \forall \bar{m} \in \mathbb{F}_2^k\}$$

Generator matrix $(k \times n)$, linearly independent rows.

… the nullspace of a matrix

$$\mathcal{C} = \{\bar{c} \in \mathbb{F}_2^n : H\bar{c}^T = \bar{0}\}$$

Parity check matrix $((n - k) \times n)$, linearly independent rows.

$$HG^{\mathrm{T}} = 0$$

Length: n, # columns in G or H

Dimension: k, # rows in G.

Minimum distance, d

Smallest Hamming distance between different codewords.

Smallest Hamming weight of non-zero codewords.

Smallest number of linearly dependent columns in H.


LINKÖPING
UNIVERSITY
www.liu.se