TMME40 Vibration Analysis of Structures
Assignment 1 (5 points)

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A common and inexpensive suspension for trailers is the leaf spring, see Fig. 1. It is a flat steel bar with a rectangular cross section pinned to the chassis at the endpoints and where the wheel axis is rigidly clamped at the midpoint.

Figure 1: A typical suspension arrangement on a trailer using a leaf spring system (`http://orusttrailer.se/IWT.html`). Note that there are two leaf springs in the picture, one for each wheel.

A simplified model of the leaf spring and wheel system is given in Fig. 2. Let the leaf spring be a simply supported beam of mass m, length L, and bending stiffness EI. In addition, assume that the wheel and the brake together have the mass $M = 2m$.

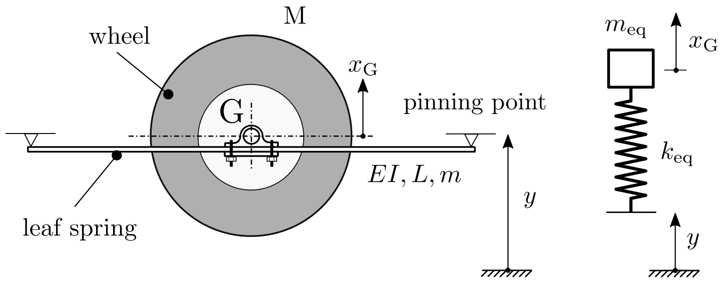


Figure 2: The simplified model for the leaf spring and wheel system (left) and the equivalent mass-spring system (right).

(a) Assume a flat road such that $y = 0$ and use beam deflection formulas in the attached table to compute the equivalent stiffness k_{eq} for the simplified leaf spring and wheel model in Fig. 2.

(b) Show that the deflection of the flat steel bar can be written

$$w(x) = w_G \frac{x}{L}\left(3 - 4\left(\frac{x}{L}\right)^2\right),$$

where $w_G = w(L/2)$.

(c) Use $w(x)$ in (b) to compute the equivalent mass m_{eq} for the whole system (flat steel bar and wheel). Also compute the natural frequency.

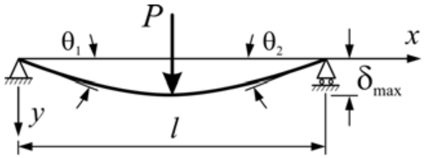
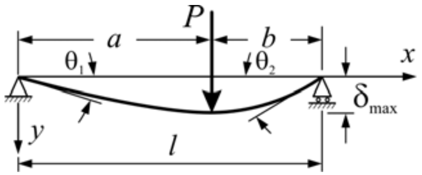
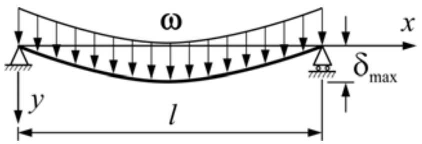
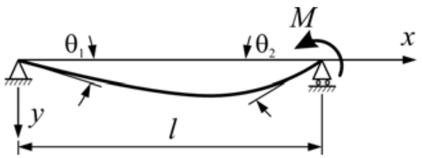
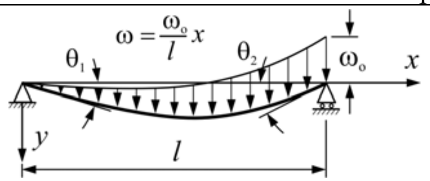
(d) Assume that the centre of the wheel is travelling with a constant velocity v_G on a flat and horizontal road. At time $t = 0$ the wheel hits a bump given by

$$y(x) = \begin{cases} Y \sin(\pi x/b), & 0 \le x \le b \\ 0, & x > b, \end{cases}$$

where Y and b are the bump height and width, respectively. Derive an expression for the motion of the wheel (point G) for $t \ge 0$. Also compute the speed v_G for which resonance occurs.

Write a short report where you present the solution to the questions above. Make sure to write your name and personal identity number (or LiU-id) at the top of each page. Hand in your printed report to the Examiner no later than 24, October 2017. Please note the rules for the exam in the course information apply.

BEAM DEFLECTION FORMULAS

BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF *x*	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load *P* at the center			
	$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$	$y = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$	$\delta_{max} = \frac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load *P* at any point			
	$\theta_1 = \frac{Pb(l^2-b^2)}{6lEI}$ $\theta_2 = \frac{Pab(2l-b)}{6lEI}$	$y = \frac{Pbx}{6lEI}\left(l^2 - x^2 - b^2\right)$ for $0 < x < a$ $y = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + \left(l^2-b^2\right)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \frac{Pb\left(l^2-b^2\right)^{3/2}}{9\sqrt{3}\,lEI}$ at $x = \sqrt{\left(l^2-b^2\right)/3}$ $\delta = \frac{Pb}{48EI}\left(3l^2 - 4b^2\right)$ at the center, if $a > b$
8. Beam Simply Supported at Ends – Uniformly distributed load ω (N/m)			
	$\theta_1 = \theta_2 = \frac{\omega l^3}{24EI}$	$y = \frac{\omega x}{24EI}\left(l^3 - 2lx^2 + x^3\right)$	$\delta_{max} = \frac{5\omega l^4}{384EI}$
9. Beam Simply Supported at Ends – Couple moment *M* at the right end			
	$\theta_1 = \frac{Ml}{6EI}$ $\theta_2 = \frac{Ml}{3EI}$	$y = \frac{Mlx}{6EI}\left(1 - \frac{x^2}{l^2}\right)$	$\delta_{max} = \frac{Ml^2}{9\sqrt{3}\,EI}$ at $x = \frac{l}{\sqrt{3}}$ $\delta = \frac{Ml^2}{16EI}$ at the center
10. Beam Simply Supported at Ends – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta_1 = \frac{7\omega_o l^3}{360EI}$ $\theta_2 = \frac{\omega_o l^3}{45EI}$	$y = \frac{\omega_o x}{360lEI}\left(7l^4 - 10l^2x^2 + 3x^4\right)$	$\delta_{max} = 0.00652\frac{\omega_o l^4}{EI}$ at $x = 0.519\,l$ $\delta = 0.00651\frac{\omega_o l^4}{EI}$ at the center

TMME40 Vibration Analysis of Structures
Assignment 2 (5 points)

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A pile driver is a mechanical device to drive piles into the soil to provide foundation support for buildings and other structures. The traditional design utilises a heavy weight which is raised above the pile and dropped. The weight slides freely down a vertical guide rail and strikes the pile. The impulse delivered at the top of the pile on impact drives it into the soil.

In this assignment you will study the axial vibrations of the pile after the impact. Assume that the pile is a steel cylinder with mass m, Young's modulus E, cross-sectional area A, and length L. The heavy weight of mass M is released a distance h above the upper end of the pile. To simplify the analysis we assume that the lower end of the pile is fixed to the support (ground) and that the heavy weight is not in contact with the pile after the impact. Data for the assignment are listed in the table at the end.

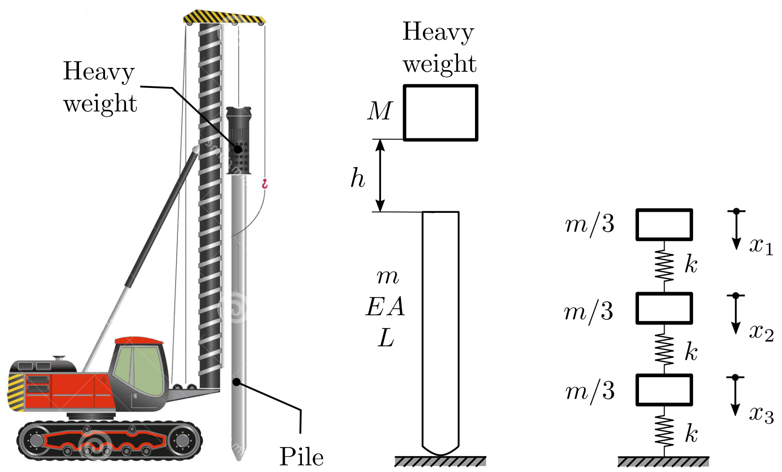

Figure 1: Left: an illustration of a pile driver (`www.thumbs.dreamstime.com`). Middle: a simplified model only including the heavy weight and the pile. Right: a three degree-of-freedom model for the pile.

(a) Use the three degree-of-freedom model for the pile and set up the equations of motion on matrix-vector form. Use free-body-diagrams and Newton's law, and remember to list the initial conditions $\vec{x}_0 = \vec{x}(0)$ and $\dot{\vec{x}}_0 = \dot{\vec{x}}(0)$. Hint: the upper mass velocity after impact $v_0 = \dot{x}_1(0^+)$ can be computed by using the coefficient of restitution and the conservation of linear momentum for the heavy weight and the upper mass together.

(b) Use Rayleigh's method and compute an estimate the pile's lowest eigenfrequency.

(c) Compute $\vec{x}(t)$ using modal analysis and sketch the mode shapes.

(d) Compute an expression for the force F_T transferred to the support (ground) and plot it for the time interval $0 \leq t \leq 0.10$ s. Estimate the maximum force during the interval.

Data for the pile driver	
Pile mass, m	250 kg
Pile area, A	25π cm^2
Pile length, L	4.0 m
Young's modulus, E	200 GPa
Heavy weight mass, M	100 kg
Drop height, h	1.0 m
Coefficient of restitution, e	0.7

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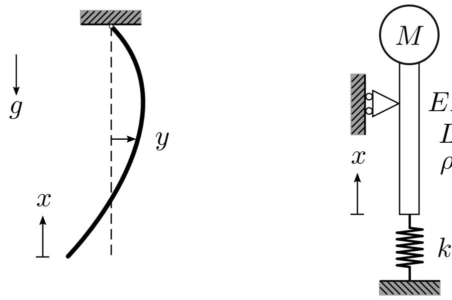

(a) (b)

(a) A flexible cable is supported at the upper end and free to oscillate under the influence of gravity in the vertical plane, see Figure (a). Under the assumption of small lateral displacements $y = y(x,t)$, show that the vibration is governed by the nonlinear partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right).$$

(b) A rod of length L, cross-section area A, density ρ and Young's modulus E is vibrating axially, see Figure (b). The lower end ($x = 0$) of the rod is supported by a spring of stiffness k_0 while the upper end ($x = L$) is attached to a mass M. Derive the boundary conditions for the system. Hint: you may wish to check your result with the boundary conditions listed in Chapter 6 in the course book.

(c) Set $M = k_0 = 0$ in question (b). First, use the separation-of-variables technique and compute the eigenfunctions $U_n(x)$ where $n = 1, 2, 3, \ldots$. Second, plot the three first eigenfunctions for $x \in [0, L]$. Finally, show that the two eigenfunctions where $n = 1$ and $n = 2$ are orthogonal with respect to the inner product

$$\int_0^L U_1(x)U_2(x)\,\mathrm{d}x.$$

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TMME40 Vibration Analysis of Structures
Solutions

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October, 2017

Assignment 1

(a) The appended beam formulas for a simply supported beam with the load P at the midpoint gives the deflection

$$w(x) = \frac{Px}{12EI}\left(\frac{3L^2}{4} - x^2\right), \quad 0 \leq x \leq \frac{L}{2}. \tag{1}$$

and the deflection at the midpoint is

$$w_{\mathrm{G}} = w(L/2) = \frac{PL^3}{48EI}. \tag{2}$$

This gives the equivalent stiffness

$$k_{\mathrm{eq}} = \frac{P}{w_{\mathrm{G}}} = \frac{48EI}{L^3}. \tag{3}$$

(b) Using Eqs. (1) and (2), the deflection can be written

$$w(x) = w_{\mathrm{G}}\frac{x}{L}\left(3 - 4\frac{x^2}{L^2}\right) \tag{4}$$

(c) The beam's kinetic energy is

$$T = \frac{1}{2}\int_0^L \frac{m}{L}\left(\dot{w}(x)\right)^2 \,\mathrm{d}x = \ldots = \frac{17}{70}m\dot{w}_{\mathrm{G}}^2, \tag{5}$$

where Eq. (4) has been used and it is assumed that $w_{\mathrm{G}} = w_{\mathrm{G}}(t)$. The kinetic energy for the equivalent system is given by

$$T = \frac{1}{2}m_{\mathrm{eq}}^{\mathrm{beam}}\dot{w}_{\mathrm{G}}^2. \tag{6}$$

Equating (5) and (6) and solving for $m_{\mathrm{eq}}^{\mathrm{beam}}$ gives

$$m_{\mathrm{eq}}^{\mathrm{beam}} = \frac{17}{35}m. \tag{7}$$

The total equivalent mass is

$$m_{\mathrm{eq}} = M + m_{\mathrm{eq}}^{\mathrm{beam}} = \ldots = \frac{87}{35}m. \tag{8}$$

A free-body diagram for the vibration around the equilibrium position for the mass gives

$$\ddot{x}_{\mathrm{G}} + \omega_{\mathrm{n}}^2 x_{\mathrm{G}} = \omega_{\mathrm{n}}^2 y, \tag{9}$$

where the natural frequency

$$\omega_{\mathrm{n}}^2 = \frac{k_{\mathrm{eq}}}{m_{\mathrm{eq}}} = \cdots = \frac{560EI}{29mL^3}. \tag{10}$$

(d) Use that $x = v_{\mathrm{G}}t$ and rewrite the 'bump equation' according to

$$y(t) = \begin{cases} Y \sin \omega t, & \text{if } 0 \leq t \leq \dfrac{b}{v_{\mathrm{G}}}, \\ 0, & \text{if } t > \dfrac{b}{v_{\mathrm{G}}}, \end{cases} \tag{11}$$

where $\omega = \pi v_{\mathrm{G}}/b$. The total solution (homogeneous and particular) to Eq. (9) reads

$$x_{\mathrm{G}}(t) = A\cos\omega t + B\sin\omega t + \frac{\omega_{\mathrm{n}}^2 Y}{\omega_{\mathrm{n}}^2 - \omega^2}\sin\omega t. \tag{12}$$

The point G is initially at rest and the initial conditions are $x_{\mathrm{G}}(0) = 0$ and $\dot{x}_{\mathrm{G}}(0) = 0$. Solving for A and B gives the solution

$$x(t) = \frac{\omega_{\mathrm{n}}^2 Y}{\omega_{\mathrm{n}}^2 - \omega^2}\left(\sin\omega t - \frac{\omega}{\omega_{\mathrm{n}}}\sin\omega_{\mathrm{n}} t\right) \tag{13}$$

Resonance when the denominator in Eq. (13) becomes zero, that is

$$\frac{\pi v_{\mathrm{G}}}{b} = \omega_{\mathrm{n}} \quad \Longrightarrow \quad v_{\mathrm{G}} = \frac{b\omega_{\mathrm{n}}}{\pi}.$$

The intention was only to solve x_G upto $t' = \frac{b}{v_{\mathrm{G}}}$, but the text may be misleading.

Assignment 2

(a) The velocity of the heavy weight just before impact is $v_{\mathrm{M}} = \sqrt{2gh}$. To compute the initial velocity v_0 for the top mass, consider the conservation of momentum for the heavy weight and the pile together

$$Mv_{\mathrm{M}} = \frac{m}{3}v_0 + Mv'_{\mathrm{M}}, \tag{1}$$

and the coefficient of restitution

$$ev_{\mathrm{M}} = v_0 - v'_{\mathrm{M}}. \tag{2}$$

Eliminating the heavy weight velocity after impact v'_{M} between the equations and solving for v_0 gives

$$v_0 = \frac{3M(1+e)}{m+3M}v_{\mathrm{M}} = \frac{3M(1+e)}{m+3M}\sqrt{2gh}. \tag{3}$$

The equivalent stiffness for the pile is $k_{\mathrm{eq}} = EA/L$. Since the springs in the model are arranged in series, we have

$$\frac{1}{k} = \frac{3}{k_{\mathrm{eq}}} \quad \Longrightarrow \quad k = 3k_{\mathrm{eq}} = \frac{3EA}{L}. \tag{4}$$

A free-body diagram for each mass gives the equation system

$$\mathbf{M}\ddot{\vec{x}} + \mathbf{K}\vec{x} = \vec{0}, \tag{5}$$

where

$$[\mathbf{M}] = \begin{bmatrix} m/3 & 0 & 0 \\ 0 & m/3 & 0 \\ 0 & 0 & m/3 \end{bmatrix}, \quad [\mathbf{K}] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}, \tag{6}$$

and

$$[\vec{x}_0] = [\vec{x}(0)] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad [\dot{\vec{x}}_0] = [\dot{\vec{x}}(0)] = \begin{pmatrix} v_0 \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

(b) Assume a force proportional to the masses, $\vec{F} \sim (1,1,1)^T$. Compute the static displacements $\vec{X} = \mathbf{A}\vec{F}$, where $\mathbf{A} = \mathbf{K}^{-1}$ is the flexibility matrix. After normalisation, we arrive at

$$\vec{X} = \begin{pmatrix} 2 \\ 5/3 \\ 1 \end{pmatrix}. \tag{8}$$

Compute the Rayleigh quotient,

$$R = \frac{\vec{X} \cdot \mathbf{K}\vec{X}}{\vec{X} \cdot \mathbf{M}\vec{X}} \approx \omega_{\mathrm{n}}^2. \tag{9}$$

Using the given values, $\omega_{\mathrm{n}} \approx 1.7 \cdot 10^3$ rad/s.

(c) Follow the algorithm in Window 4.5 in Inman. The result is

$$\vec{x}(t) = \begin{pmatrix} -0.00133 \\ -0.00107 \\ -0.00059 \end{pmatrix} \sin(1673.33t) + \begin{pmatrix} -0.00031 \\ 0.00017 \\ 0.00038 \end{pmatrix} \sin(4688.57t) + \begin{pmatrix} -0.00007 \\ 0.00015 \\ -0.00012 \end{pmatrix} \sin(6775.18t)\,[\mathrm{m}]. \quad (10)$$

The eigenvectors $\vec{v}_i$ $(i = 1, 2, 3)$ computed from the mass-normalised stiffness matrix $\tilde{\mathbf{K}} = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$, where $\mathbf{M} = \mathbf{L}\mathbf{L}^T$, can be used to compute the mode shapes $\vec{u}_i$. The eigenvectors and the mode shapes are related through $\vec{u}_i = \mathbf{L}^{-T}\vec{v}_i$. Note that the eigenvector computation can introduce a (very) small imaginary part due to numerics. In these cases, only use the real part of $\vec{v}_i$.

(d) The force transmitted to the ground is given by

$$F_{\mathrm{T}} = kx_3. \quad (11)$$

The maximum transmitted force during the interval is $F_{\mathrm{T}}^{\max} \approx 1200$ kN.

Assignment 3

(a) Let τ be the stress in the cable at position x and take the cable's cross-sectional area be A and the density ρ. A free-body diagram (must be drawn!) for a small part of the cable $\mathrm{d}x$ at x gives the two equations of motion:

$$\uparrow:\ (\tau + \mathrm{d}\tau)A\cos(\theta + \mathrm{d}\theta) - \tau A\cos\theta - \rho g A\,\mathrm{d}x = 0, \tag{1}$$

$$\rightarrow:\ -\tau A\sin\theta + (\tau + \mathrm{d}\tau)A\sin(\theta + \mathrm{d}\theta) = \rho g A\ddot{y}\,\mathrm{d}x. \tag{2}$$

For small angles, $\cos(\theta) \approx 1$ and Eq. (1) becomes

$$\frac{\mathrm{d}\tau}{\mathrm{d}x} = -\rho q, \tag{3}$$

and, by integration,

$$\tau(x) = -\rho g x. \tag{4}$$

Rewriting Eq. (2) using that for small angles $\sin\theta \approx \theta$ and neglecting the higher-order term $\mathrm{d}\tau\,\mathrm{d}\theta$ gives

$$\tau\frac{\mathrm{d}\theta}{\mathrm{d}x} + \frac{\mathrm{d}\tau}{\mathrm{d}x}\theta = \rho\ddot{y}. \tag{5}$$

Finally, by substituting Eqs. (3) and (4) into (5) and using that $\theta \approx \tan\theta = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, we obtain

$$\ddot{y} + g\frac{\partial y}{\partial x} + xg\frac{\partial^2 y}{\partial x^2} = \ddot{y} + g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) \text{ QED.}$$

(b) A free-body diagram (must be drawn!) of a small slice with thickness Δx at the left end gives the equation of motion

$$\uparrow:\ -ku(0,t) + EA\frac{\partial u}{\partial x}(0,t) = \rho A\ddot{u}(0,t)\Delta x.$$

Since $\Delta x \to 0$, we have

$$EA\frac{\partial u}{\partial x}(0,t) = ku(0,t).$$

For the right end, we draw free-body diagrams for both the rod and the mass. The equation of motion for the mass is

$$\uparrow:\ -N = M\ddot{u}(L,t), \tag{6}$$

where N is the contact force between the rod and the mass. For the right end of the rod, the equation of motion reads

$$\uparrow:\ -EA\frac{\partial u}{\partial x}(L,t) + N = \rho A\ddot{u}(L,t)\Delta x. \tag{7}$$

Letting $\Delta x \to 0$ and eliminating N between the two equations (6) and (7) gives

$$-EA\frac{\partial u}{\partial x}(L,t) = M\ddot{u}(L,t).$$

(c) For $k = M = 0$ we have a free-free rod:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \\ \dfrac{\partial u}{\partial x}(0,t) = 0, \quad \dfrac{\partial u}{\partial x}(L,t) = 0. \end{cases} \tag{8}$$

The separation-of-variables ansatz

$$u(x,t) = U(x)T(t)$$

gives

$$\frac{U''}{U} = \frac{\ddot{T}}{T} = -k^2, \tag{9}$$

where $(\cdot)'' = \partial^2(\cdot)/\partial x^2$. Solving the spatial problem in Eq. $(9)_1$ gives

$$U(x) = C_1 \cos kx + C_2 \sin kx. \tag{10}$$

Since the boundary conditions in Eq. $(8)_{2,3}$ must hold for all time t, we have

$$\frac{\partial U}{\partial x}(0) = kC_2 = 0, \tag{11}$$

and

$$\frac{\partial U}{\partial x}(L) = kC_1 \sin kL - kC_2 \cos kL = 0. \tag{12}$$

Equations (11) and (12) give $C_2 = 0$ and the characteristic equation $\sin kL = 0$, which has the solutions

$$kL = n\pi, \quad n = 0, 1, 2, \ldots \tag{13}$$

Thus, the eigenfunctions are given by

$$U_n(x) = \cos\left(\frac{n\pi}{L}x\right). \tag{14}$$

Finally, the orthogonality of of the eigenfunctions $U1(x)$ and $U_2(x)$ is shown by computing the inner product

$$\begin{aligned} \int_0^L \cos\left(\frac{\pi}{L}x\right)\cos\left(\frac{2\pi}{L}x\right)\,\mathrm{d}x &= \int_0^L \left[\cos\left(\frac{\pi}{L}x\right) + \cos\left(\frac{3\pi}{L}x\right)\right]\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{L}{\pi}\sin\left(\frac{\pi}{L}x\right) + \frac{L}{3\pi}\sin\left(\frac{3\pi}{L}x\right)\right]_0^L = 0, \end{aligned} \tag{15}$$

where the trigonometric identity

$$2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$$

has been used in the first equality.