

TSDT14 Signal Theory

Lecture 5

Prediction and Non-Linear Systems

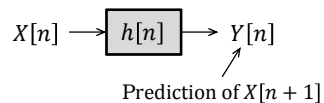
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Prediction

WSS TD process $X[n]$.

Task: Predict next sample.

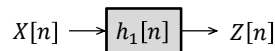
Method: Filter the signal.



Prediction error:

$$Z[n] = Y[n] - X[n+1].$$

Interpretation:



Prediction error filter:

$$h_1[n] = h[n] - \delta[n+1]$$

$$H_1[\theta] = H[\theta] - e^{j2\pi\theta}$$

Mean: $m_Z = (H[0] - 1)m_X$

Unbiased: $m_Z = 0$

$$\Rightarrow H[0] = 1 \text{ or } m_X = 0$$

Quality measure (distorsion):

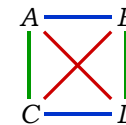
$$\varepsilon^2 = r_Z[0] = (h_1 * \tilde{h}_1 * r_X)[0]$$

$$= \int_0^1 |H_1[\theta]|^2 R_X[\theta] d\theta$$

$$\tilde{h}_1[n] = h_1[-n]$$

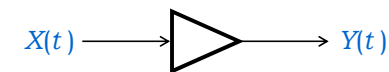
3- & 4-Dim. Jointly Gaussian Variables

$$E\{ABCD\} = E\{AB\} \cdot E\{CD\} + E\{AC\} \cdot E\{BD\} + E\{AD\} \cdot E\{BC\} \\ - 2 \cdot E\{A\} \cdot E\{B\} \cdot E\{C\} \cdot E\{D\}$$



$$E\{ABC\} = E\{AB\} \cdot E\{C\} + E\{AC\} \cdot E\{B\} + E\{A\} \cdot E\{BC\} \\ - 2 \cdot E\{A\} \cdot E\{B\} \cdot E\{C\}$$

Example: Amplifier 1(3)



Input: $X(t)$ stationary Gaussian process with $m_X = 0$.

Output: $Y(t) = X(t) + \varepsilon X^2(t)$. Quadratic distorsion.

Mean: $m_Y = E\{X(t) + \varepsilon X^2(t)\} = \varepsilon \cdot E\{X^2(t)\} = \varepsilon \cdot r_X(0)$

$\uparrow$
 $m_X = 0$

ACF: $r_Y(\tau) = E\{Y(t)Y(t+\tau)\} = E\{(X(t) + \varepsilon X^2(t))(X(t+\tau) + \varepsilon X^2(t+\tau))\}$

$$= \underbrace{E\{X(t)X(t+\tau)\}}_{r_X(\tau)} + \underbrace{\varepsilon \cdot E\{X^2(t)X(t+\tau)\}}_{=?} + \underbrace{\varepsilon \cdot E\{X(t)X^2(t+\tau)\}}_{=?} + \underbrace{\varepsilon^2 \cdot E\{X^2(t)X^2(t+\tau)\}}_{=?}$$

Example: Amplifier 2(3)

$$\begin{aligned} E\{X^2(t)X(t+\tau)\} &= \\ &= E\{X^2(t)\} \cdot E\{X(t+\tau)\} + 2E\{X(t)\} \cdot E\{X(t)X(t+\tau)\} - 2E^2\{X(t)\} \cdot E\{X(t+\tau)\} \\ &= r_X(0) \cdot m_X + 2 \cdot m_X \cdot r_X(\tau) - 2 \cdot m_X^3 = m_X \cdot (r_X(0) + r_X(\tau) - 2m_X^2) = 0 \end{aligned}$$

$$E\{X(t)X^2(t+\tau)\} = \dots = m_X \cdot (r_X(0) + r_X(\tau) - 2m_X^2) = 0 \quad \leftarrow \text{Since } m_X=0$$

$$\begin{aligned} E\{X^2(t)X^2(t+\tau)\} &= \\ &= E\{X^2(t)\} \cdot E\{X^2(t+\tau)\} + 2E^2\{X(t)X(t+\tau)\} - 2E^2\{X(t)\} \cdot E^2\{X(t+\tau)\} \\ &= r_X^2(0) + 2 \cdot r_X^2(\tau) - 2 \cdot m_X^4 = r_X^2(0) + 2 \cdot r_X^2(\tau) \quad \leftarrow \text{Since } m_X=0 \end{aligned}$$

$$\begin{aligned} \text{Totally: } r_Y(\tau) &= r_X(\tau) + \varepsilon^2 \cdot (r_X^2(0) + 2 \cdot r_X^2(\tau)) \\ R_Y(f) &= R_X(f) + \varepsilon^2 \cdot (r_X^2(0) \cdot \delta(f) + 2 \cdot (R_X * R_X)(f)) \end{aligned}$$

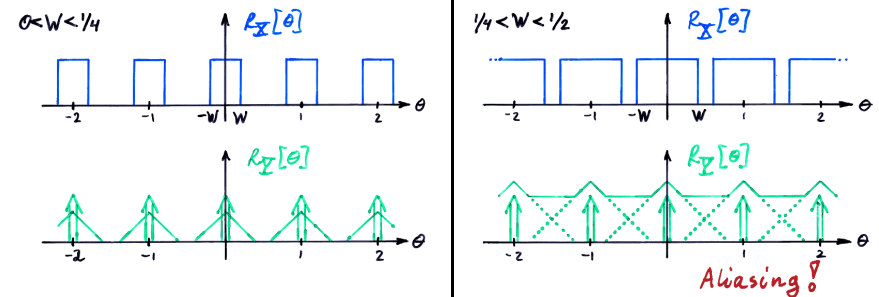
Squaring a Time-Discrete Gaussian Process

Input: $X[n]$ stationary Gaussian with $m_X=0$. (Bandlimited)

$$\text{Output: } Y[n] = X^2[n] \quad r_Y[k] = r_X^2[0] + 2r_X^2[k]$$

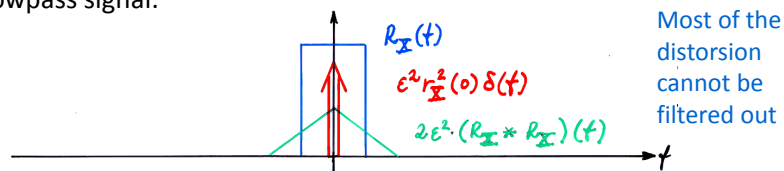
$$R_Y[\theta] = r_X^2[0] \cdot \sum_m \delta(\theta - m) + 2(R_X \otimes R_X)[\theta]$$

$$\text{Periodic convolution: } (R_X \otimes R_X)[\theta] = \int_0^1 R_X[\phi] R_X[\theta - \phi] d\phi$$

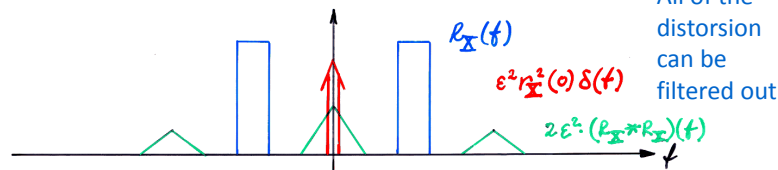


Example: Amplifier 3(3)

Lowpass signal:



Bandpass signal:



Price's Theorem

Useful to determine the ACF after a nonlinear mapping of Gaussian processes.

Prerequisites: (A, B) is jointly Gaussian with mean $(0, 0)$, $f(a)$ and $g(b)$ are functions, usually nonlinear, and $\rho = E\{AB\}$.

$$\text{Then we have: } \frac{\partial^n}{\partial \rho^n} E\{f(A)g(B)\} = E\left\{\frac{\partial^n}{\partial A^n} f(A) \frac{\partial^n}{\partial B^n} g(B)\right\}$$

Example Price's Theorem 1(2)

Assumptions: $\mathbf{X}(t)$ strictly stationary Gaussian Process with $m_{\mathbf{X}}=0$.
 $\mathbf{Y}(t) = \mathbf{X}^2(t)$

Objective: Express $r_{\mathbf{Y}}(\tau)$ in $r_{\mathbf{X}}(\tau)$.

Method: Price's theorem with $A=\mathbf{X}(t)$, $B=\mathbf{X}(t+\tau)$,
 $f(a)=g(a)=a^2$ and $S=r_{\mathbf{X}}(\tau)$.

We have $r_{\mathbf{Y}}(\tau) = E\{\mathbf{Y}(t)\mathbf{Y}(t+\tau)\} = E\{\mathbf{X}^2(t)\mathbf{X}^2(t+\tau)\} = E\{f(\mathbf{X}(t)) \cdot g(\mathbf{X}(t+\tau))\}$

Price's theorem for $n=1$ gives us

$$\frac{\partial}{\partial r_{\mathbf{X}}(\tau)} r_{\mathbf{Y}}(\tau) = E\left\{\frac{\partial}{\partial A} f(A) \cdot \frac{\partial}{\partial B} g(B)\right\} = E\left\{\frac{\partial A^2}{\partial A} \cdot \frac{\partial B^2}{\partial B}\right\} = E\{2A \cdot 2B\} \\ = 4E\{\mathbf{X}(t)\mathbf{X}(t+\tau)\} = 4r_{\mathbf{X}}(\tau)$$

This gives us

$$\partial r_{\mathbf{Y}}(\tau) = 4r_{\mathbf{X}}(\tau) \cdot \partial r_{\mathbf{X}}(\tau) \Rightarrow \int \partial r_{\mathbf{Y}}(\tau) = \int 4r_{\mathbf{X}}(\tau) \partial r_{\mathbf{X}}(\tau) \Rightarrow \\ \Rightarrow r_{\mathbf{Y}}(\tau) + C_1 = 2r_{\mathbf{X}}^2(\tau) + C_2 \Rightarrow r_{\mathbf{Y}}(\tau) = 2r_{\mathbf{X}}^2(\tau) + C$$

More Non-Linearities

From
Tables & Formulas,
Page 15.

Situation

The input to a momentary non-linearity is a Gaussian process $X(t)$ with mean $m_X = 0$ and auto-correlation function $r_X(\tau)$.

Identities

$Y(t)$	$r_Y(\tau)$
$X^2(t)$	$2r_X^2(\tau) + r_X^2(0)$
$X^3(t)$	$6r_X^3(\tau) + 9r_X^2(0)r_X(\tau)$
$X^4(t)$	$24r_X^4(\tau) + 72r_X^2(0)r_X^2(\tau) + 9r_X^4(0)$
$X^5(t)$	$120r_X^5(\tau) + 600r_X^3(0)r_X^3(\tau) + 225r_X^4(0)r_X(\tau)$
$\begin{cases} X, & X \geq 0, \\ 0, & X < 0. \end{cases}$	$\frac{r_X(\tau)}{4} + \frac{1}{2\pi} \left[\sqrt{r_X^2(0) - r_X^2(\tau)} + r_X(\tau) \arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right) \right]$
	$= \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^3(\tau)}{4\pi r_X(0)} + \dots$
$\text{sgn}(X)$	$\frac{2}{\pi} \arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right)$

The corresponding
relations hold for time-
discrete processes as
well.

Example Price's Theorem 2(2)

We had: $r_{\mathbf{Y}}(\tau) = 2r_{\mathbf{X}}^2(\tau) + C$

Objective: Determine the constant C .

How? Choose a τ such that $r_{\mathbf{X}}(\tau) = 0$ holds.

Then: $\mathbf{X}(t)$ & $\mathbf{X}(t+\tau)$ are uncorrelated. $\Rightarrow \mathbf{X}(t)$ & $\mathbf{X}(t+\tau)$ indep.

But: $\mathbf{X}(t)$ & $\mathbf{X}(t+\tau)$ jointly Gaussian

So: $C = r_{\mathbf{Y}}(\tau) = E\{\mathbf{X}^2(t)\mathbf{X}^2(t+\tau)\} = E\{\mathbf{X}^2(t)\} \cdot E\{\mathbf{X}^2(t+\tau)\} = r_{\mathbf{X}}^2(0)$
 $\uparrow$ $\uparrow$
 $r_{\mathbf{X}}(\tau)=0$ Indep. $r_{\mathbf{X}}(0)$ $r_{\mathbf{X}}(0)$

Totally: $r_{\mathbf{Y}}(\tau) = 2r_{\mathbf{X}}^2(\tau) + r_{\mathbf{X}}^2(0)$

Half-Wave Rectifier

From the table:

$$\begin{cases} X, & X \geq 0, \\ 0, & X < 0. \end{cases} \quad \left| \begin{aligned} & \frac{r_X(\tau)}{4} + \frac{1}{2\pi} \left[\sqrt{r_X^2(0) - r_X^2(\tau)} + r_X(\tau) \arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right) \right] \\ & = \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^3(\tau)}{4\pi r_X(0)} + \dots \end{aligned} \right|$$

Complete Maclaurin expansion:

$$r_Y(\tau) = \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^3(\tau)}{4\pi r_X(0)} + \sum_{n=2}^{\infty} \frac{(2n-3)!! \cdot r_X(0)}{2\pi \cdot (2n-1) \cdot (2n)!!} \left(\frac{r_X(\tau)}{r_X(0)}\right)^{2n}$$

$n!!$ is semi-factorial (product of every second positive integer):

$$6!! = 2 \cdot 4 \cdot 6 = 48$$

and

$$7!! = 1 \cdot 3 \cdot 5 \cdot 7 = 105.$$

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