TSKS02 Telecommunication

Lecture 2
Linear Filters in the Frequency Domain

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The Principle sine in – sine out

For LTI systems, we have:

Sine in – Sine out (the same frequency)

Compare to particular solution of a differential equation.

Also: the $j\omega$ method.

Linearity implies:

Σ Sine in – Σ Sine out

Desireable:

Describe signals in terms of sines.

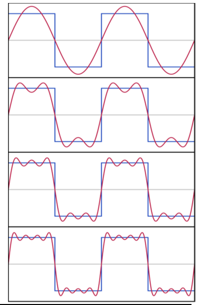
Thus: Fourier series and Fourier transforms

Fourier Series Expansion

Jean Baptiste Joseph Fourier
1768 – 1830

Demands on the signal $x(t)$:

1. Periodic, period T.
2. Absolute integrable: $\int_0^T |x(t)|\,dt < \infty$
3. A finite number of local min & max in a period.
4. A finite number of discontinuities in a period.



Then $X_0, \hat{X}_k, \varphi_k$ exist for $k \in \{1, 2, \ldots, \infty\}$ such that

$$x(t) = X_0 + \sum_{k=1}^{\infty} \underbrace{\hat{X}_k \sin(k 2\pi f_1 t + \varphi_k)}_{\text{Tone number } k.}$$

holds with $f_1 = 1/T$.

Complex Fourier Series Expansion

Euler: $$\sin(k2\pi f_1 t + \varphi_k) = \frac{e^{j(k2\pi f_1 t + \varphi_k)} - e^{-j(k2\pi f_1 t + \varphi_k)}}{j2}$$

Result: $$x(t) = X_0 + \sum_{k=1}^{\infty} \hat{X}_k \sin(k2\pi f_1 t + \varphi_k)$$

$$= \underbrace{X_0}_{C_0} + \sum_{k=1}^{\infty} \left(\underbrace{\frac{\hat{X}_k e^{j\varphi_k}}{j2}}_{C_k} e^{jk2\pi f_1 t} - \underbrace{\frac{\hat{X}_k e^{-j\varphi_k}}{j2}}_{C_{-k}} e^{-jk2\pi f_1 t} \right) = \sum_{k=-\infty}^{\infty} C_k e^{jk2\pi f_1 t}$$

Relations:

$X_0 = C_0$ $\qquad$ $\varphi_k = \frac{\pi}{2} + \arg(C_k), \quad \forall\, k > 0$

$\hat{X}_k = 2|C_k|, \quad \forall\, k > 0$ $\qquad$ $C_{-k} = C_k^*, \quad \forall\, k$

Determining C_k

Consider:

$$\int_0^T x(t)\, e^{-jk2\pi f_1 t} dt = \int_0^T \sum_{m=-\infty}^{\infty} C_m e^{jm2\pi f_1 t} e^{-jk2\pi f_1 t} dt = \sum_{m=-\infty}^{\infty} C_m \underbrace{\int_0^T e^{j(m-k)2\pi f_1 t} dt}_{=\begin{cases}0, & m\neq k\\ T, & m=k\end{cases}} = TC_k$$

Thus:

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk2\pi f_1 t} dt$$

But also:

$$C_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jk2\pi f_1 t} dt$$

Reason:

$$x(t+T) = x(t)$$

$$e^{-jk2\pi f_1(t+T)} = e^{-jk2\pi(f_1 t+1)} = e^{-jk2\pi f_1 t}$$

Properties of Fourier Series Expansions

Let $x(t)$ and $y(t)$ be periodic signals with period T and Fourier series coefficients C_k and D_k, respectively.

Signal	Fourier series coefficient # k
$ax(t)+by(t)$	$aC_k + bD_k$
$x(t-\tau)$	$C_k e^{-jk2\pi f_1 \tau}$
$x(at)$	C_k (period T/a)
$\frac{d}{dt}x(t)$	$jk2\pi f_1 C_k$

Fourier Transform

Demands on the signal $x(t)$:

- Absolute integrable: $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$
- Finite number of discontinuities.
- Limited variation: $\int_{-\infty}^{\infty} |x'(t)|\, dt < \infty$

Jean Baptiste Joseph Fourier
1768 – 1830

Transform:

$$X(f) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt$$

Inverse transform:

$$x(t) = \mathscr{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df$$

- Spectrum of $x(t)$: $X(f)$
- Amplitude spectrum: $|X(f)|$
- Phase spectrum: $\arg\{X(f)\}$

Signal Power and Signal Energy – Parseval

Signal power: $|x(t)|^2$

Signal energy: $\int_{-\infty}^{\infty} |x(t)|^2\, dt$

Parseval's relation (special case):

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t)x^*(t)\, dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)e^{j2\pi f t}\, df\; x^*(t)\, dt =$$

$$= \int_{-\infty}^{\infty} X(f)\int_{-\infty}^{\infty} x^*(t)e^{j2\pi f t}\, dt\, df = \int_{-\infty}^{\infty} X(f)X^*(f)\, df = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

Energy spectrum: $|X(f)|^2$

Parseval's relation (generally): $\int_{-\infty}^{\infty} a(t)b^*(t)\, dt = \int_{-\infty}^{\infty} A(f)B^*(f)\, df$

Output from an LTI System

Notation: $A(f)=\mathscr{F}\{a(t)\}$ $\qquad B(f)=\mathscr{F}\{b(t)\}$

Property: $\mathscr{F}\{(a*b)(t)\}=\int_{-\infty}^{\infty}(a*b)(t)\,e^{-j2\pi f t}\,dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a(\tau)b(t-\tau)\,d\tau\, e^{-j2\pi f t}\,dt$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a(\tau)b(t-\tau)e^{-j2\pi f t}\,d\tau\,dt=\Bigg/\begin{matrix}\lambda=t-\tau\\ d\lambda=dt\end{matrix}\Bigg/=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a(\tau)b(\lambda)e^{-j2\pi f(\tau+\lambda)}\,d\tau\,d\lambda$$

$$=\int_{-\infty}^{\infty}a(\tau)e^{-j2\pi f\tau}\,d\tau\int_{-\infty}^{\infty}b(\lambda)e^{-j2\pi f\lambda}\,d\lambda=A(f)B(f)$$

LTI System:

$$\begin{matrix}x(t)\\X(f)\end{matrix}\longrightarrow\boxed{\begin{matrix}h(t)\\H(f)\end{matrix}}\longrightarrow\begin{matrix}y(t)=(x*h)(t)\\Y(f)=X(f)H(f)\end{matrix}$$

Periodic signals

Observation: $\mathscr{F}^{-1}\{\delta(f-f_0)\}=\int_{-\infty}^{\infty}\delta(f-f_0)\,e^{j2\pi f t}\,dt\ =e^{j2\pi f_0 t}$

Thus: $\mathscr{F}\left\{e^{j2\pi f_0 t}\right\}=\delta(f-f_0)$

Euler: $\mathscr{F}\{\cos(2\pi f_0 t)\}=\mathscr{F}\left\{\dfrac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}\right\}=\dfrac{1}{2}\left(\delta(f-f_0)+\delta(f+f_0)\right)$

$$\mathscr{F}\{\sin(2\pi f_0 t)\}=\mathscr{F}\left\{\frac{e^{j2\pi f_0 t}-e^{-j2\pi f_0 t}}{j2}\right\}=\frac{1}{2j}\left(\delta(f-f_0)-\delta(f+f_0)\right)$$

$x(t)$ periodic with period T: $\qquad f_1=\dfrac{1}{T}$

$$x(t)=\sum_{k=-\infty}^{\infty}C_k e^{jk2\pi f_1 t}\quad\Rightarrow\quad X(f)=\sum_{k=-\infty}^{\infty}C_k\delta(f-kf_1)$$

The Principle sine in – sine out

For LTI systems, we have:

Sine in – Sine out (the same frequency)

More precisely:

Input: $x(t)=\hat{X}\sin(2\pi f_0 t+\varphi)$

Output: $y(t)=\hat{X}\,|H(f_0)|\sin\left(2\pi f_0 t+\varphi+\arg\{H(f_0)\}\right)$

Amplitude characteristic: $|H(f)|$

Phase characteristic: $\arg\{H(f)\}$

This is the $j\omega$ method in condensed form.

Classification of Frequency Selective Filters

- A (frequency selective) filter is an LTI system. Usually it lets some frequency band through or stops it.
- Usually an electrical network – either passive or active.

Notation | Ideal amplitude characteristics | Real amplitude char.

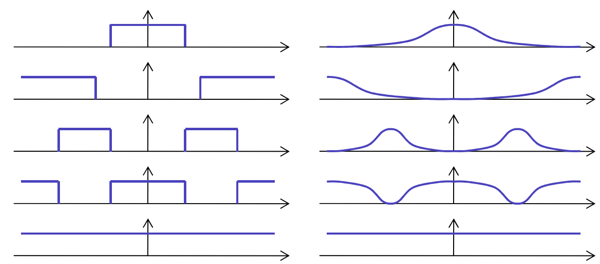
Lowpass filter (LP filter):

Highpass filter (HP filter):

Bandpass filter (BP filter):

Bandstop filter (BS filter):

Allpass filter (AP filter):



Important Property: Derivatives

Notation: $X(f) = \mathscr{F}\{x(t)\}$

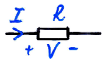
We have: $$\frac{d}{dt}x(t) = \frac{d}{dt}\int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$
$$= \int_{-\infty}^{\infty} X(f)\frac{d}{dt}e^{j2\pi f t}\, df = \int_{-\infty}^{\infty} \underbrace{X(f)\, j2\pi f}_{\mathscr{F}\left\{\frac{d}{dt}x(t)\right\}}\, e^{j2\pi f t}\, df$$

Result: $$\mathscr{F}\left\{\frac{d}{dt}x(t)\right\} = j2\pi f X(f)$$
$$\mathscr{F}\left\{\frac{d^k}{dt^k}x(t)\right\} = (j2\pi f)^k X(f)$$

Laws of Electricity 1(2)

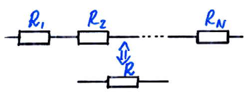
Resistance

Definition: (Ohm's law)



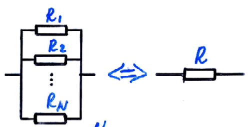
$V = R \cdot I$

In series:



$R = \sum_{k=1}^{N} R_k$

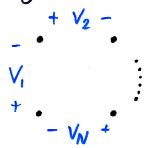
In parallel:



$\frac{1}{R} = \sum_{k=1}^{N} \frac{1}{R_k}$

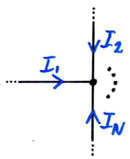
Kirchhoff's laws

The voltage law:



$\sum_{k=1}^{N} V_k = 0$

The current law:

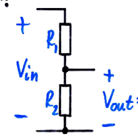


$\sum_{k=1}^{N} I_k = 0$

Laws of Electricity 2(2)

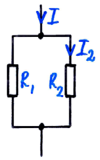
Simple Circuits

Voltage divider:



$V_{out} = \frac{R_2}{R_1+R_2} V_{in}$

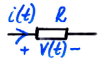
Current divider:



$I_2 = \frac{R_1}{R_1+R_2} \cdot I$

General voltages & currents

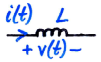
Resistance:



$Z_R = R$

$v(t) = R \cdot i(t)$ $\quad V(f) = R \cdot I(f)$

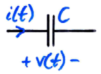
Inductance:



$Z_L = j2\pi f L$

$v(t) = L \cdot \frac{d}{dt} i(t)$ $\quad V(f) = j2\pi f L \cdot I(f)$

Capacitance:



$Z_C = \frac{1}{j2\pi f C}$

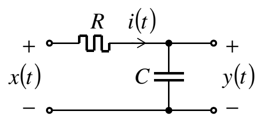
$i(t) = C \frac{d}{dt} v(t)$ $\quad I(f) = j2\pi f C \cdot V(f)$

$\Rightarrow V(f) = \frac{1}{j2\pi f C} \cdot I(f)$

Important Property: Derivatives Again

Notation: $X(f) = \mathscr{F}\{x(t)\}$ We have: $\mathscr{F}\left\{\frac{d}{dt}x(t)\right\} = j2\pi f\, X(f)$

Example:



Energy free $y(t)$ initially 0.

Capacitance: $i(t) = C\frac{d}{dt}y(t)$ (1)

Resistance: $x(t) - y(t) = Ri(t)$ (2)

(1) in (2) $\Rightarrow x(t) - y(t) = RC\frac{d}{dt}y(t)$

$\Rightarrow RC\frac{d}{dt}y(t) + y(t) = x(t)$

Transform $\Rightarrow$

$j2\pi f\, RCY(f) + Y(f) = X(f)$

$\Rightarrow (j2\pi f\, RC + 1)Y(f) = X(f)$

$\Rightarrow Y(f) = \frac{1}{j2\pi f\, RC + 1} X(f)$

Transform and inverse transform usually using a table.

Time-Discrete Signals and Systems

Unit impulse: $\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$ Impulse response: $h[n]$

Unit step: $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$ Step response: $g[n]$

Convolution: $y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

Fourier transform: $X[\theta] = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi\theta n}$

Inverse: $x[n] = \mathcal{F}^{-1}\{X[\theta]\} = \int_0^1 X[\theta]\, e^{j2\pi\theta n}\, d\theta$

Periodicity: $X[\theta] = X[\theta + k]$, k integer.

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Transformations

Sampling

Time domain: $\tilde{x}[n] = x(nT)$

Freq. domain: $\tilde{X}[\theta] = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\theta - k}{T}\right)$

Sampling freq: $f_s = \frac{1}{T}$

Sampling period: T

Pulse-Amplitude Modulation (PAM)

Time domain: $y(t) = \sum_{n=-\infty}^{\infty} x[n]\, p(t - nT)$

Freq. domain: $Y(f) = X[fT] \cdot P(f)$

Normalized freq: $\theta = fT$