TSKS01 Digital Communication
Lecture 6

Detection: Bits and pieces for AWGN channels and Dispersive channels

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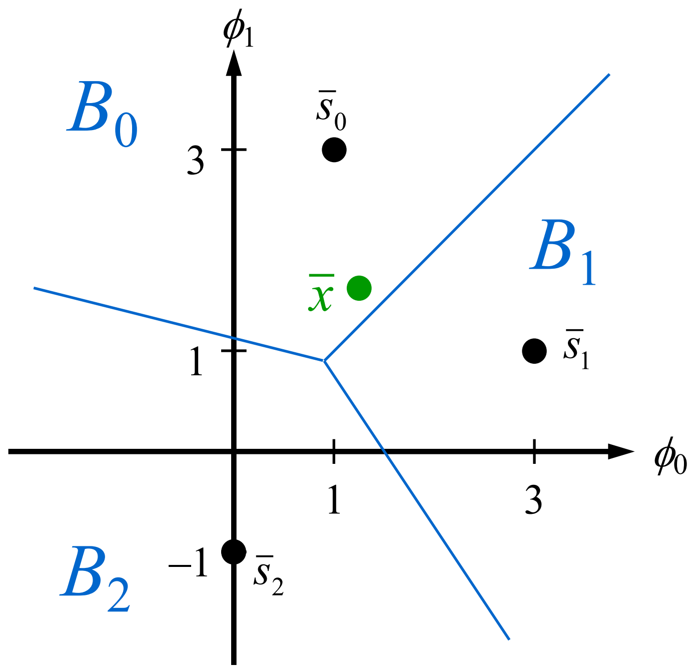
What remains: Bits and pieces

- More bounds and approximations of P_e
- Cost of non-optimal detection
- Detection of individual bits

Recall: ML Detection Bounds



ML Decision rule:
Set $\hat{a} = a_i$ if $d(\bar{x}, \bar{s}_k)$ is minimized for $k = i$.

Decision region:
All points closest to a signal.

Error probability:

$$P_e = \frac{1}{M} \sum_{i=0}^{M-1} \Pr\left\{ \bar{X} \notin B_i \mid A = a_i \right\}$$

Union bound:

Over-estimation!

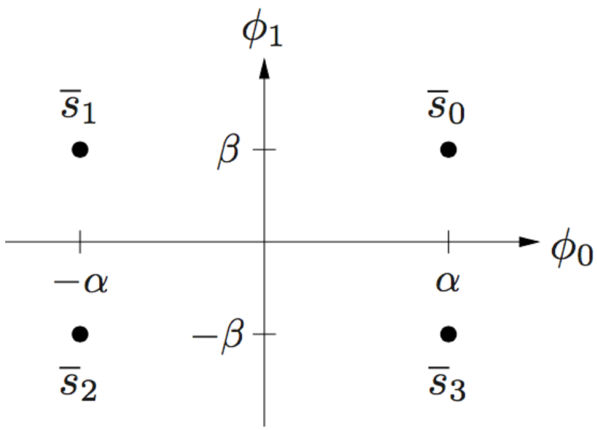
$$P_e \le \frac{1}{M} \sum_{i=0}^{M-1} \sum_{j \ne i} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}} \right)$$

Nearest neighbour approximation:

$$P_e \approx \frac{1}{M} \sum_{i=0}^{M-1} n_i Q\left(\frac{d_{\min}}{\sqrt{2N_0}} \right)$$

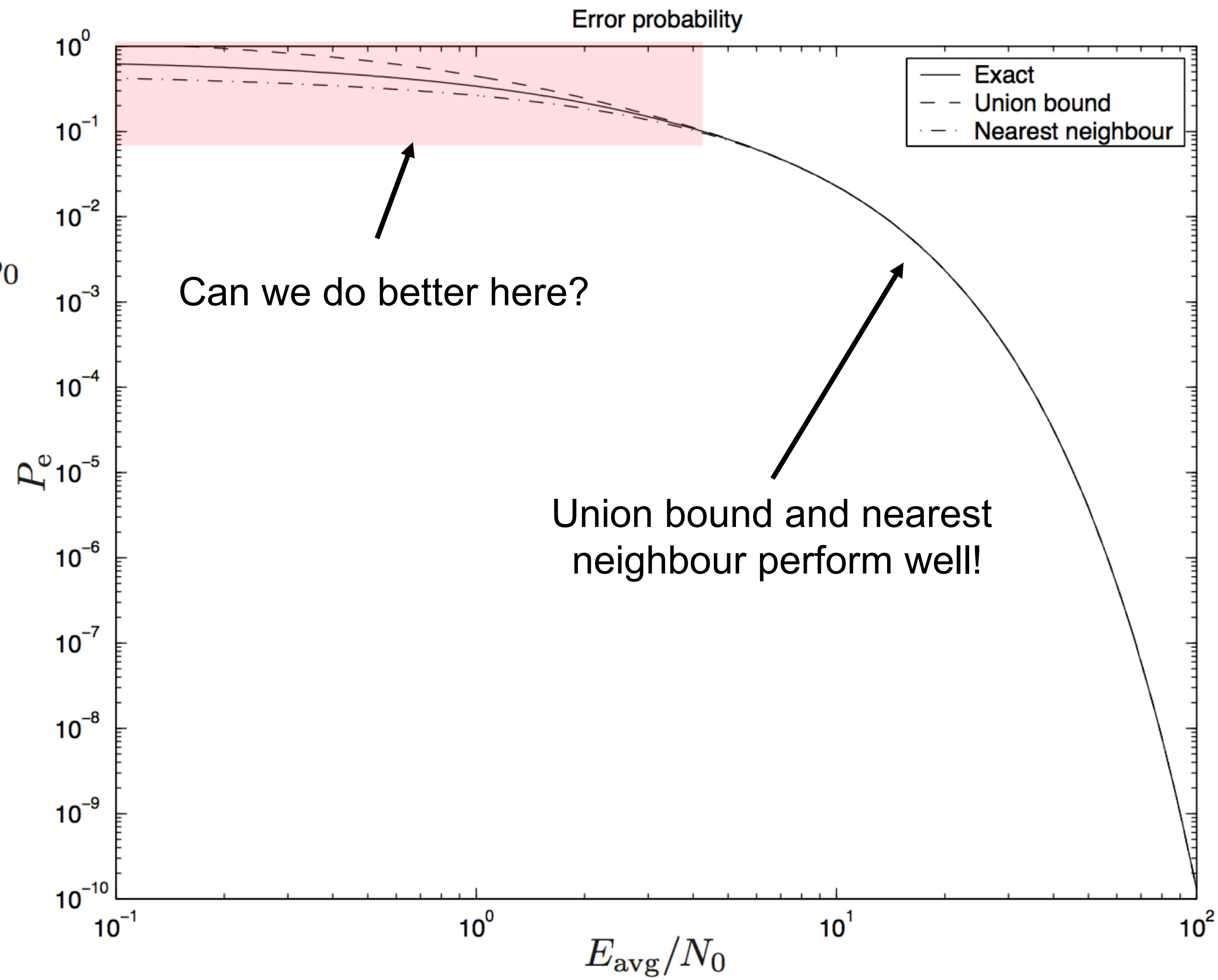
Average # nearest neighbors

Example: Error Probability

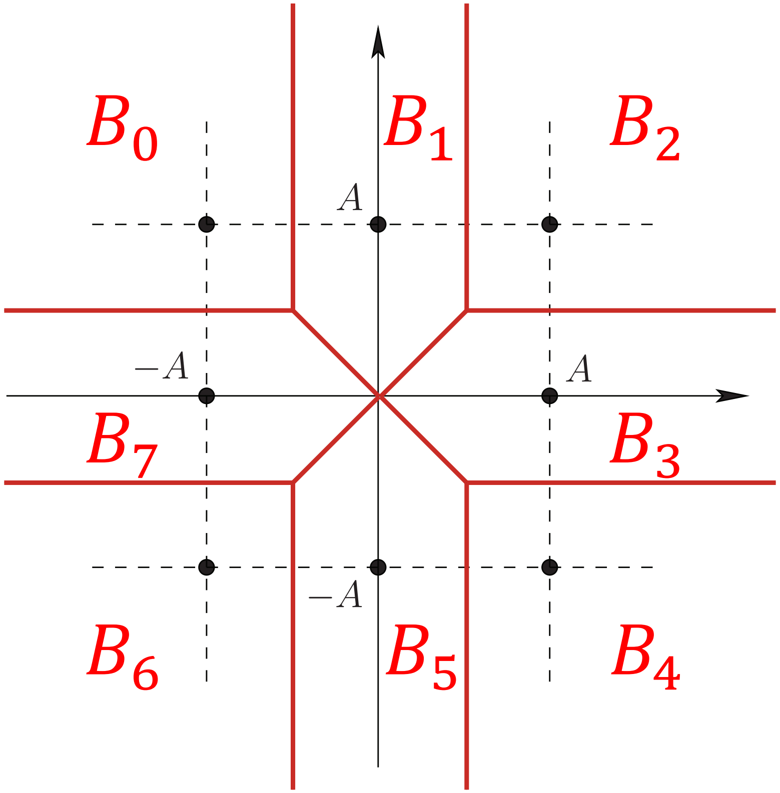


Assume:
$\alpha = 2\beta$

$$E_{\mathrm{avg}} = \alpha^2 + \beta^2 = 5\beta^2$$



Alternative Union Bound



$M = 8, N = 2$

Many points with small N:

Some regions have no common border!

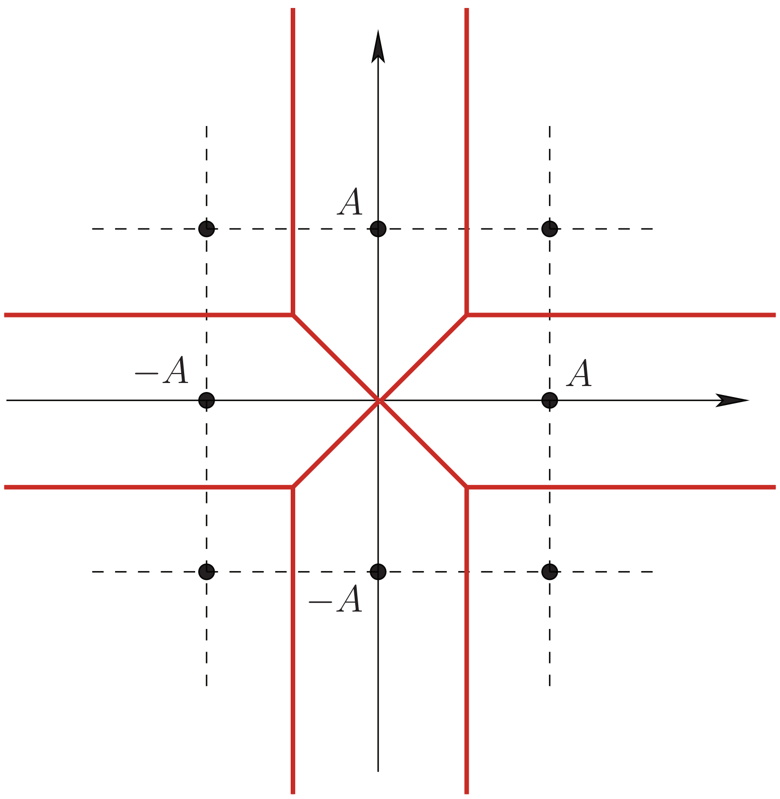
Set of "real" neighbours:

$$I_j = \{i\!: i \neq j \text{ and } B_i, B_j \text{ have a common border}\}$$

Improved union bound:

$$P_e \leq \frac{1}{M} \sum_{j=0}^{M-1} \sum_{i \in I_j} Q\left(\frac{d_{ji}}{\sqrt{2N_0}}\right)$$

Alternative Nearest Neighbour Approximation



$d_1 = A, \qquad d_2 = \sqrt{2}A, \qquad d_3 = 2A$

Order distances between points:

$$d_1 < d_2 < d_3 < \cdots$$

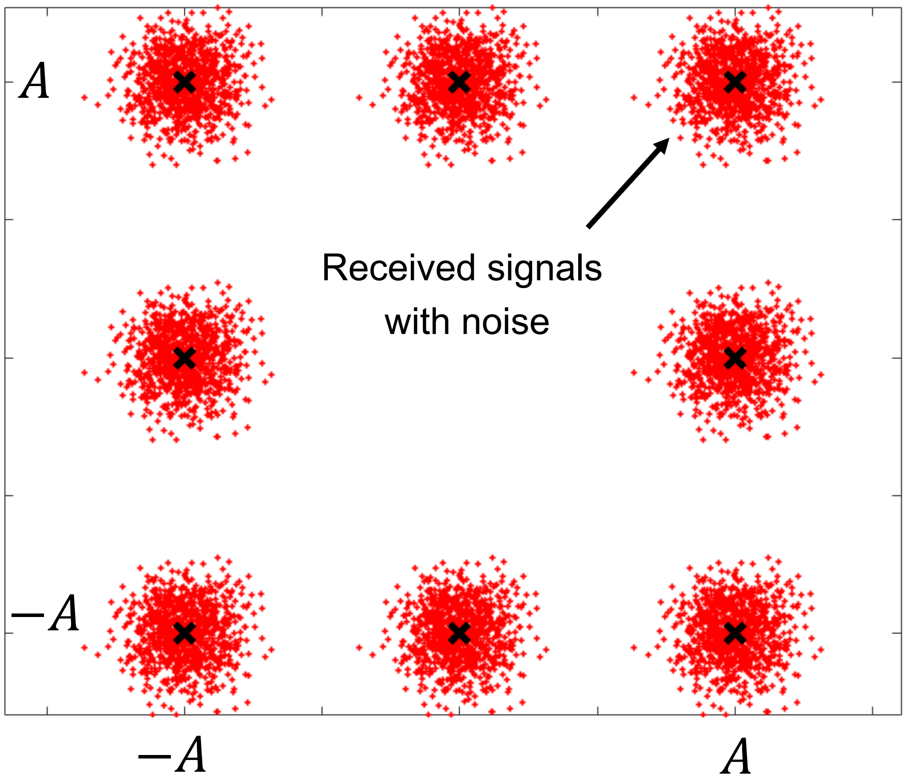
Count neighbours at different distances:

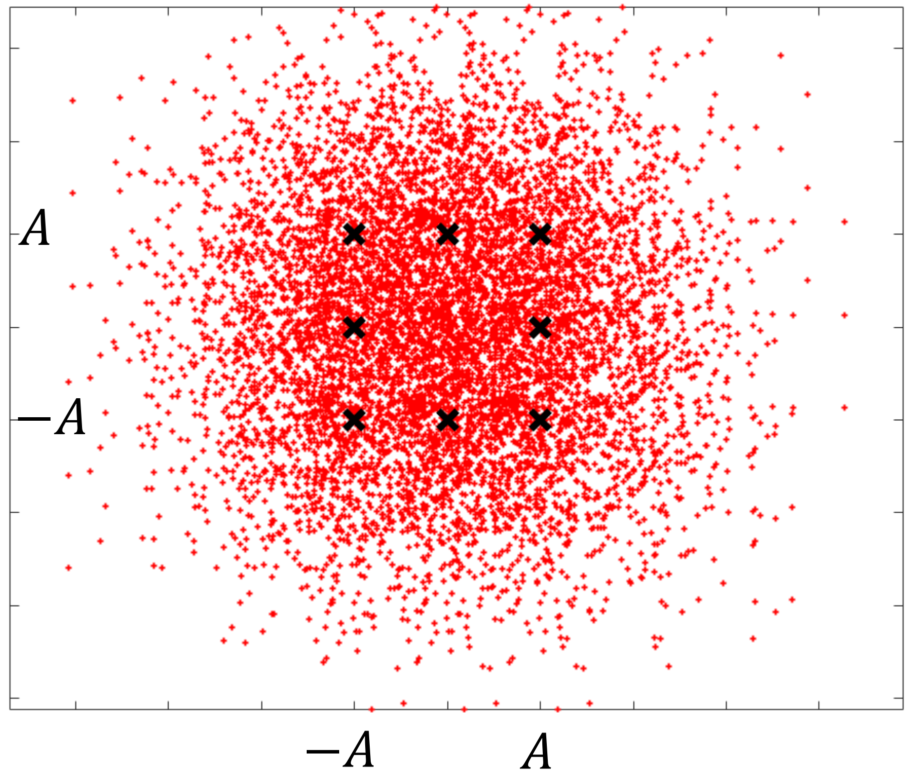
n_{ij} = # of neighbours at distance d_j from $\bar{s}_i$

NN approximation with m distances:

$$P_e \approx \sum_{j=1}^{m} Q\left(\frac{d_j}{\sqrt{2N_0}}\right) \frac{1}{M} \sum_{i=0}^{M-1} n_{ij}$$

Example: Received Signals

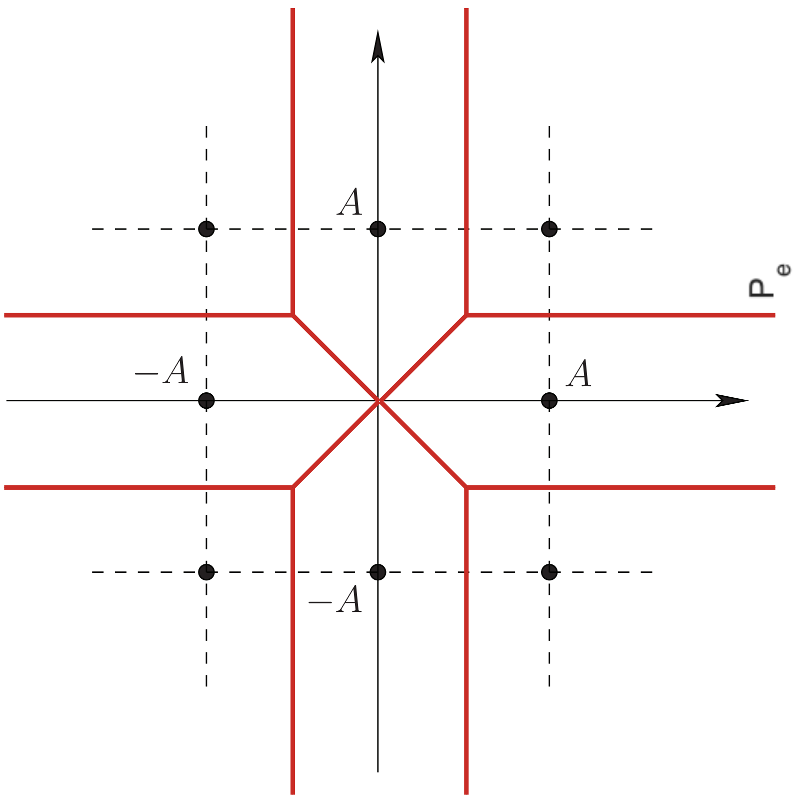


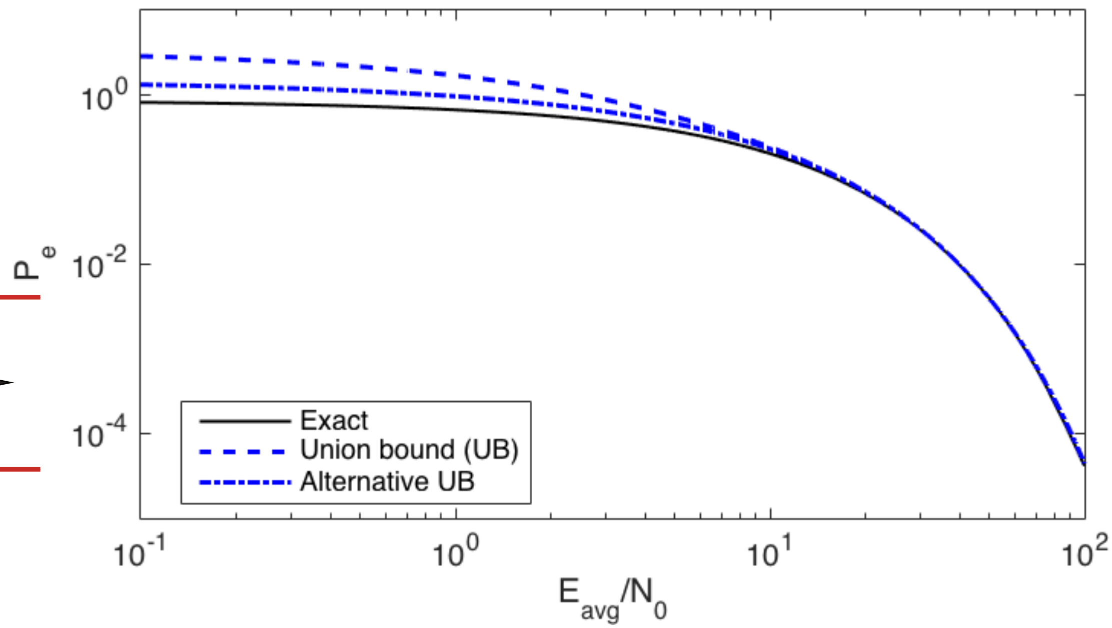


SNR: $E_{\text{avg}}/N_0 = 7.5$

SNR: $E_{\text{avg}}/N_0 = 0.75$

Example: Alternative Union Bound





Alternative bound is better at low SNRs

Simple but crude expressions (1/2)

- Rough approximation:

$$P_e \approx Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

  - Each point has one nearest neighbour!
  - Small N: Within an order of magnitude of correct value
  - $N = 2$: No signal point can have more than 6 nearest neighbours

- Simple upper bound:

$$P_e \leq (M-1)Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

  - Replace all distances in UB with d_{min}: Further over-estimation!

Simple but crude expressions (2/2)

- A lower bound:
  - $k =$ # signals with at least one nearest neighbour
  - Lower bound for these signals is $Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$
  - Ignore all other error events:

$$P_e \geq \frac{k}{M} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

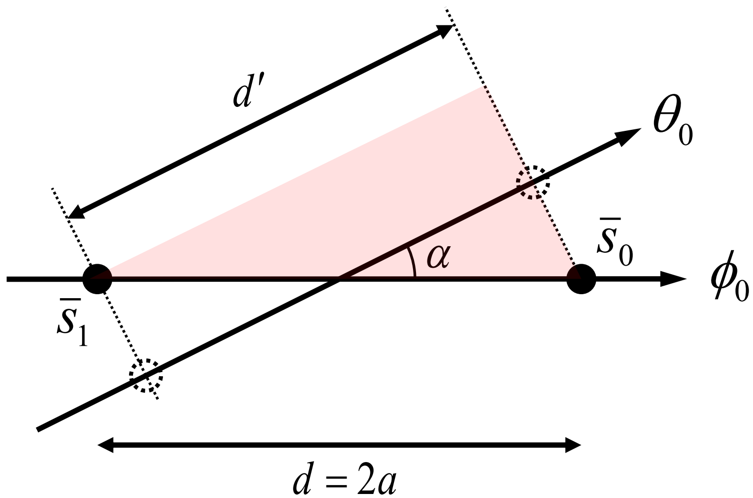
- Crude lower bound ($k \geq 2$):

$$P_e \geq \frac{2}{M} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Comparison:

$(M-1)/(\frac{2}{M}) \approx M^2/2$ for large M and $(M-1)/(\frac{2}{M}) = 1$ for $M = 2$

Cost of Non-Optimal Detection



Signals: $s_0(t) = a \cdot \phi_0(t)$

$s_1(t) = -a \cdot \phi_0(t)$

Wrong basis function in receiver:

$$\theta_0(t)$$

What is the cost of this non-ideal situation?

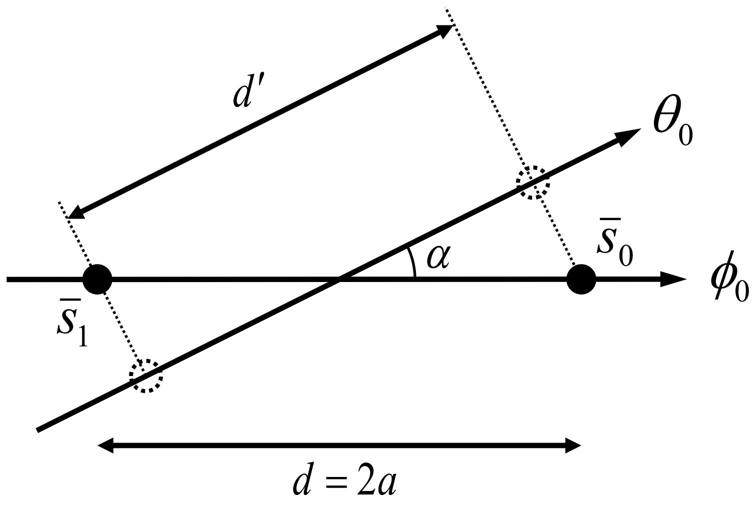
Detected signal points: $(\pm s_0, \theta_0) = \pm a \cdot (\phi_0, \theta_0) = \pm a \cdot \|\phi_0\| \cdot \|\theta_0\| \cdot \cos(\alpha) = \pm a \cdot \cos(\alpha)$

Effective distance: $d' = d \cdot \cos(\alpha)$

Resulting error probability: $P_e = Q\left(\dfrac{d'}{\sqrt{2N_0}}\right)$

Signals are projected on basis functions

Example of Non-Optimal Detection



SNR: $\frac{a^2}{N_0} = 5$

Signal points: $\pm a$

Distances: $d = 2a$

$d' = d \cdot \cos(\alpha)$

Ideal detection: $P_e = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\sqrt{10}\right) \approx Q(3.16) \approx 7.9 \cdot 10^{-4}$

16×

At $\alpha = \pi/4$: $P_e = Q\left(\frac{d'}{\sqrt{2N_0}}\right) = Q\left(\sqrt{5}\right) \approx Q(2.24) \approx 1.25 \cdot 10^{-2}$

Detection of Individual Bits

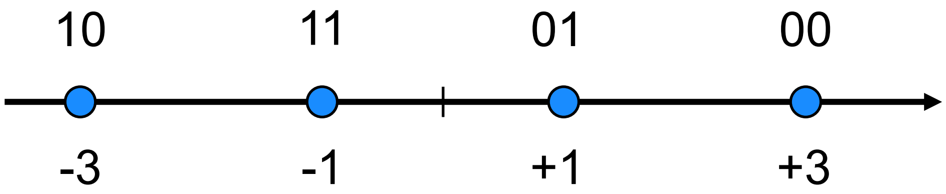
Question:

Is ML detection of symbols, followed by mapping to bits actually ML detection of bits?

What happens if we detect each bit directly?

Example: 4-ASK

Bits:



B_0 is the first bit, B_1 is the second bit

Assume equal probability

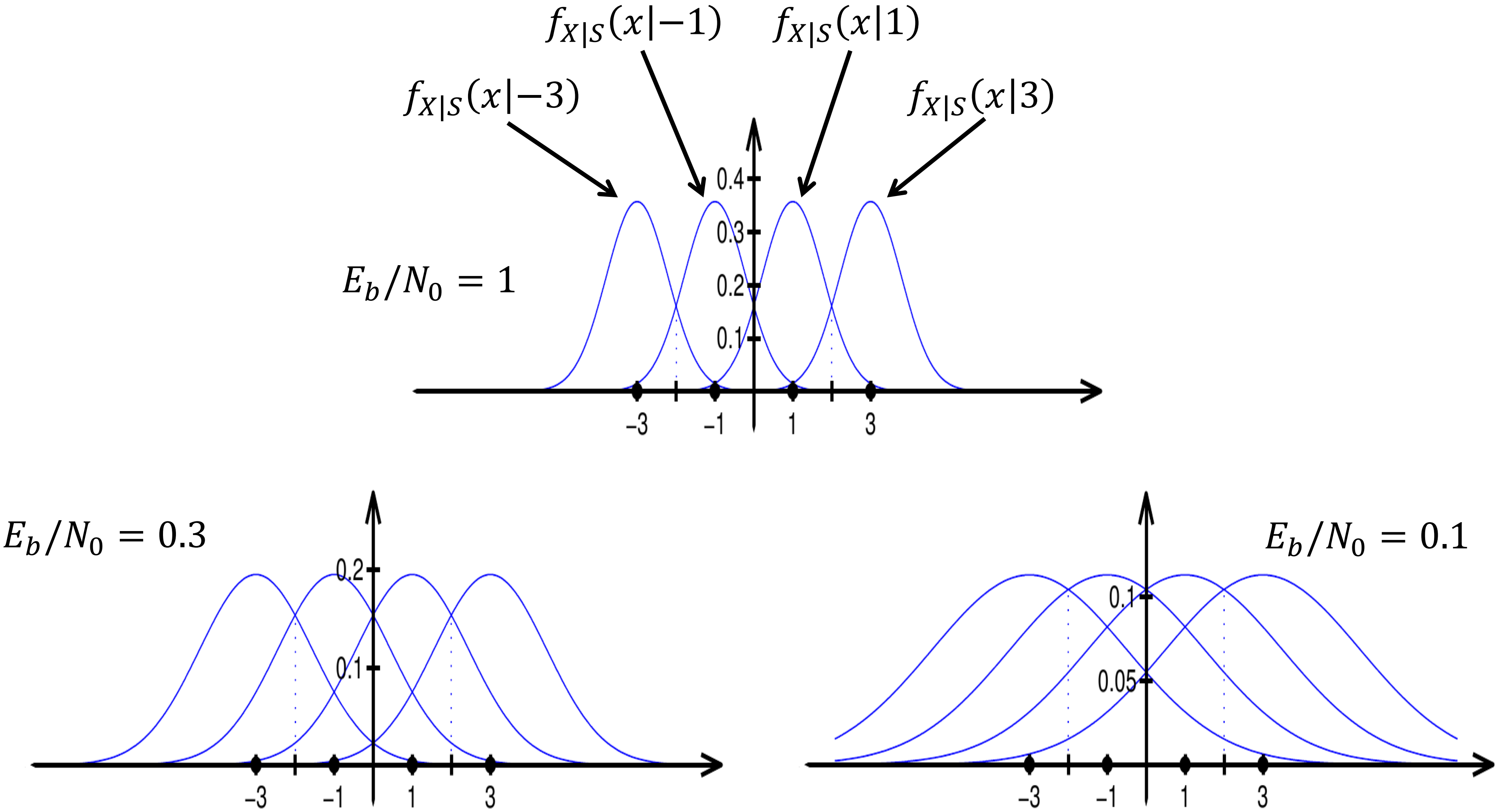
Notation:

S : Signal point as stochastic variable (follows from B_0 and B_1)

X : Received value as stochastic variable

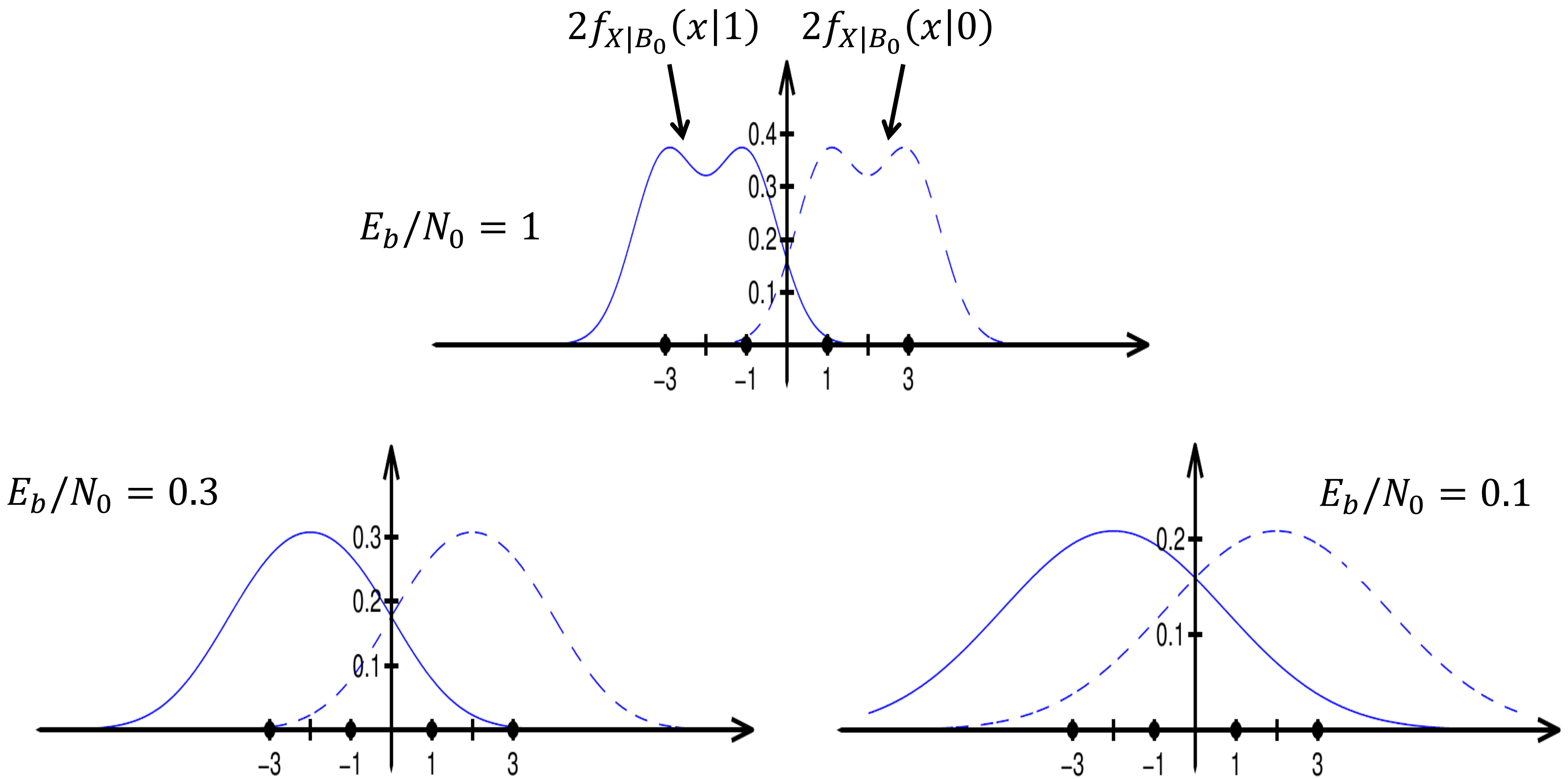
x : Realization of X

ML Detection of Symbols



Decision borders: –2, 0, 2

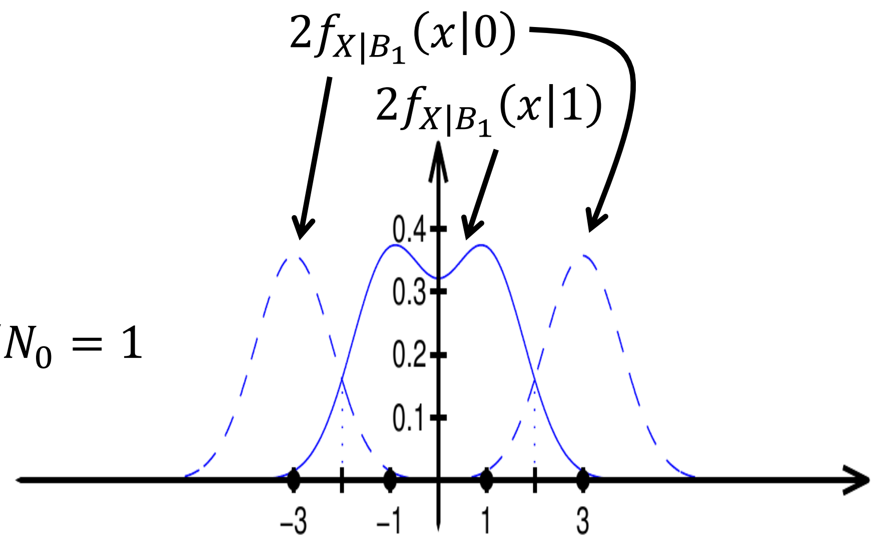
ML Detection of B_0



Decision border: 0

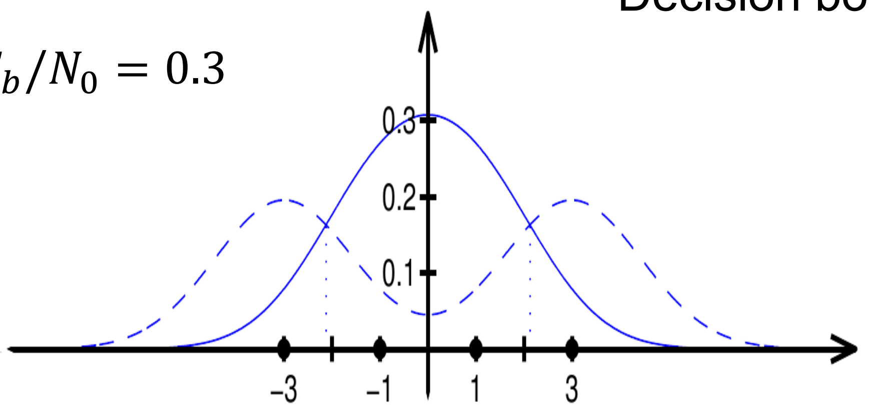
ML Detection of B_1

$E_b/N_0 = 1$



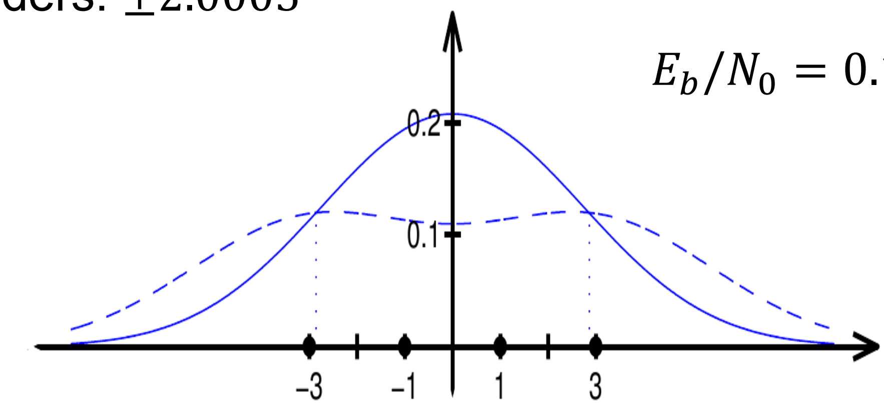
Decision borders: ± 2.0005

$E_b/N_0 = 0.3$



Decision borders: ± 2.125

$E_b/N_0 = 0.1$



Decision borders: ± 2.858

Detection of Individual Bits: Summary

Question:

Is ML detection of symbols, followed by mapping to bits actually ML detection of bits?

Answer:

It depends on the constellation and on the mapping of bits to symbols:

- *4-ASK: Some difference at low SNR ($E_b/N_0 < 1$)*
- *QPSK: No difference at all!*

How should we detect?

Independent bits: Detect each bit separately

Dependent bits: Detect a whole sequence (symbol by symbol – or more!)

Recall: Nyquist Criterion

Recall: Pulse-Amplitude Modulation (PAM):

$$x(t) = \sum_{n} s[n]p(t - nT)$$

Receiver filtering $\gamma(t), \Gamma(f)$:

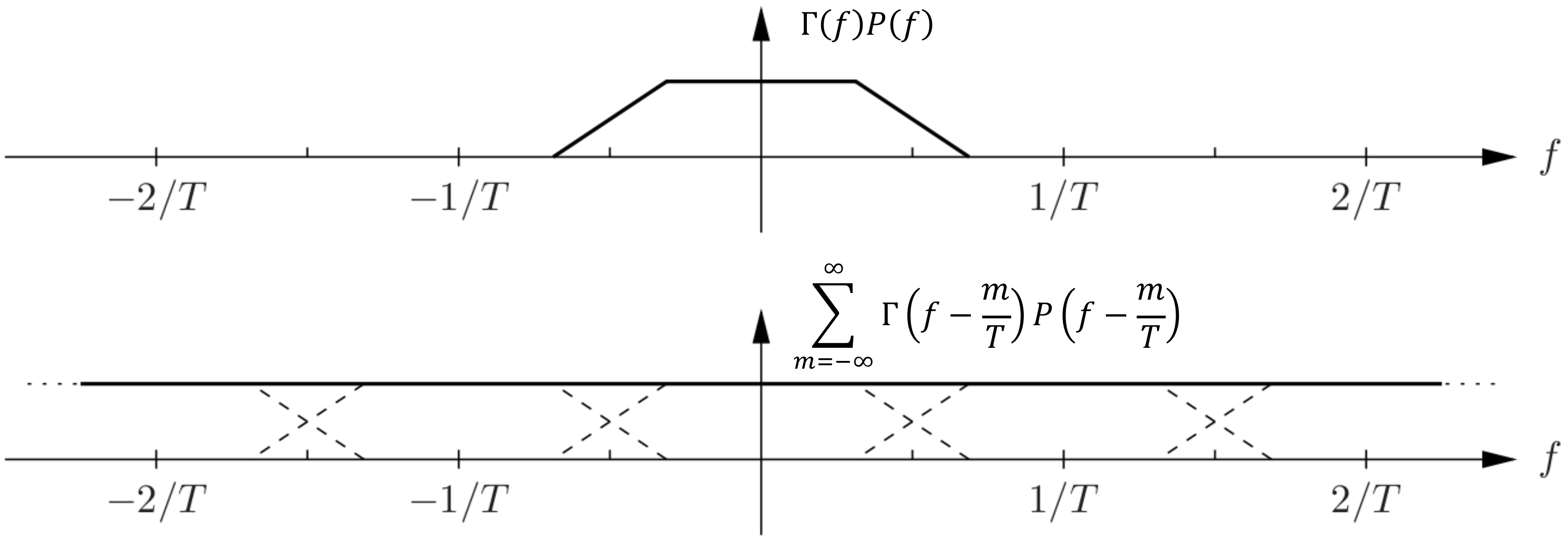
$$(\gamma * x)(t) = \sum_{n} s[n](\gamma * p)(t - nT)$$

Nyquist criterion:

$$\sum_{m=-\infty}^{\infty} \Gamma\left(f - \frac{m}{T}\right) P\left(f - \frac{m}{T}\right) = \text{Constant}$$

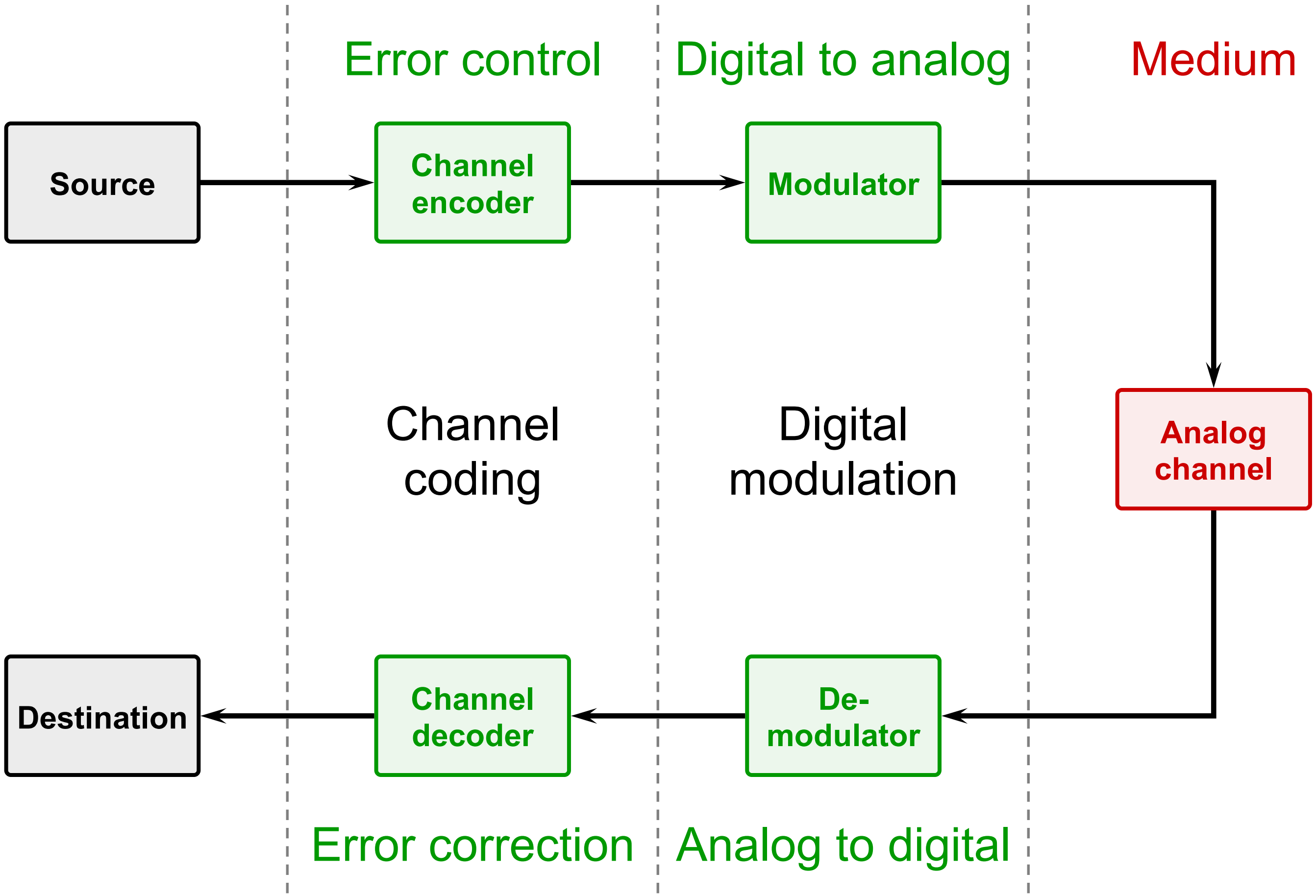
Called: $p(t)$ and $\gamma(t)$ are **Nyquist together**

Bandwidth more than $1/(2T)$: Many Nyquist pulses



$p(t)$ and $\gamma(t)$ are **Nyquist together** if $\sum_{m=-\infty}^{\infty} \Gamma\left(f-\frac{m}{T}\right) P\left(f-\frac{m}{T}\right)$ is constant

One-way Digital Communication System

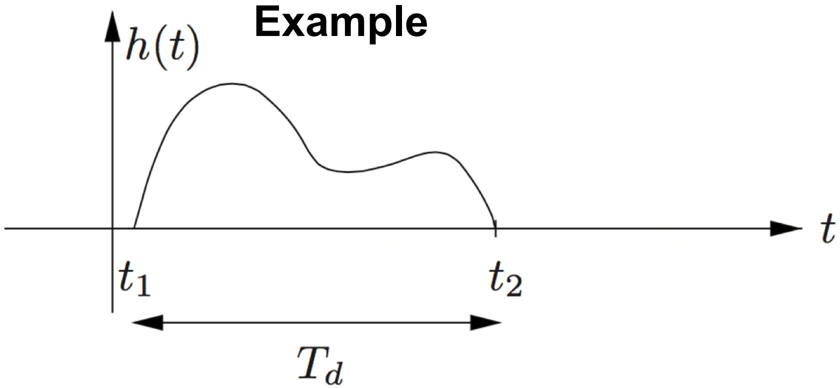


Channel properties

Impulse response $h(t)$

Additive noise $W(t)$

Channel Filtering



- Transmitted PAM signal

$$S(t) = \sum_n s[n]p(t - nT)$$

  - Filtered by channel, impulse response $h(t)$:

$$S(t) \longrightarrow \boxed{h(t)} \longrightarrow (S * h)(t)$$

  - Noise is added

$$X(t) = (S * h)(t) + W(t)$$

How will this affect the Nyquist criterion?

Nyquist Criterion with Channel Filtering (1/2)

Signal after Receiver filtering $\gamma(t), \Gamma(f)$:

$$(\gamma * x)(t) = \sum_n s[n](\gamma * p * h)(t - nT) + (\gamma * W)(t)$$

Effective pulse:

$$(\gamma * p * h)(t)$$

Nyquist criterion:

$$\sum_{m=-\infty}^{\infty} \Gamma\left(f - \frac{m}{T}\right) H\left(f - \frac{m}{T}\right) P\left(f - \frac{m}{T}\right) = \text{Constant}$$

Can this be achieved in practice?

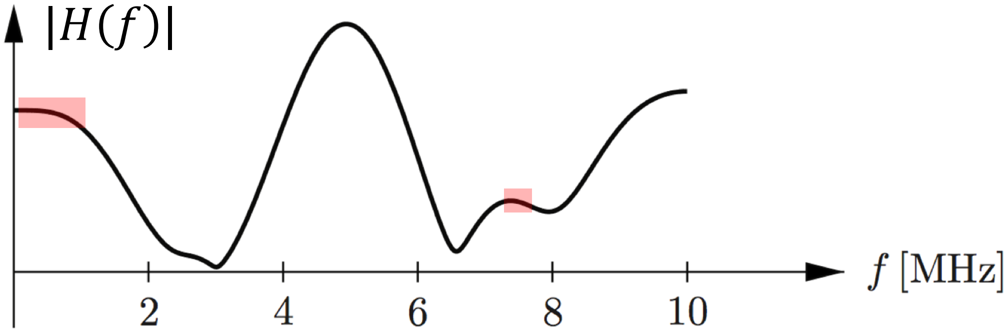
Nyquist Criterion with Channel Filtering (2/2)

Nyquist criterion $\sum_{m=-\infty}^{\infty} \Gamma\left(f-\frac{m}{T}\right) H\left(f-\frac{m}{T}\right) P\left(f-\frac{m}{T}\right)$ is constant

- If $H(f) \neq 0$ everywhere with $\mathrm{P}(f) \neq 0$ and $\mathrm{P}(f)$ is Nyquist
  - Pick $\Gamma(f) = 1/H(f)$ Impractical to depend on $H(f)$!

- If $H(f) \approx \text{constant}$ where $\mathrm{P}(f) \neq 0$
  - Pick $\Gamma(f)$ and $P(f)$ as Nyquist together

Example:



Small bandwidth:
Approximately constant
One reason to use OFDM

What if the Nyquist Criterion is not satisfied?

- Sampled output

$$Z[k] = \sum_n s[n](\gamma * p * h)(kT - nT) + \underbrace{(\gamma * W)(kT)}_{W_\gamma[k]}$$

  - Define $g[l] = (\gamma * p * h)(lT)$

- Inter-symbol interference if $g[l] \neq 0$ for some $l \neq 0$
  - Causal and time-limited pulses and impulse response
  - $g[l] \neq 0$ for $l = 0, \dots, L-1$

$$Z[k] = \sum_{l=0}^{L-1} s[k-l]g[l] + W_\gamma[k]$$

L-tap Channel with Gaussian Noise

- Sampled output

$$Z[k] = \sum_{l=0}^{L-1} s[k-l]g[l] + W_\gamma[k]$$

where $W_\gamma[k]$ is

- Gaussian with zero mean and variance $N_0/2$
- Independent for different values of k

This is called a dispersive channel

How to perform detection? This is the topic of the next lecture

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