

TSDT14 Signal Theory

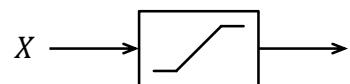
Lecture 6

Saturation, Quantization and Poisson Processes

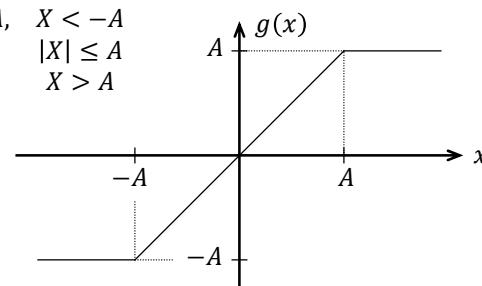
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Saturation

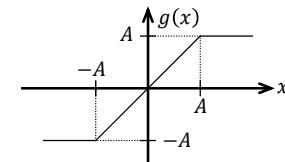


$$Y = g(X) = \begin{cases} -A, & X < -A \\ X, & |X| \leq A \\ A, & X > A \end{cases}$$



Saturation PDF

$$Y = g(X) = \begin{cases} -A, & X < -A \\ X, & |X| \leq A \\ A, & X > A \end{cases}$$



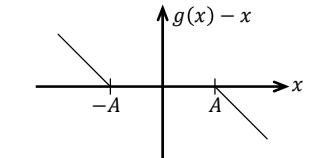
$$p_1 = \Pr\{X < -A\} = \int_{-\infty}^{-A} f_X(x) dx \quad p_2 = \Pr\{X > A\} = \int_A^{\infty} f_X(x) dx$$

$$\tilde{f}_Y(y) = f_X(y) \cdot \text{rect}\left(\frac{y}{2A}\right) = \begin{cases} f_X(y), & |y| \leq A \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y) = p_1 \delta(y + A) + \tilde{f}_Y(y) + p_2 \delta(y - A)$$

Saturation Error

$$S = Y - X = g(X) - X = \begin{cases} -A - X, & X < -A \\ 0, & |X| \leq A \\ A - X, & X > A \end{cases}$$

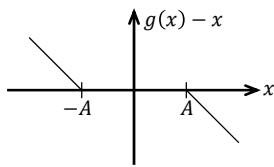


$$p_3 = \Pr\{|X| \leq A\} = \int_{-A}^A f_X(x) dx = 1 - p_1 - p_2$$

$$\tilde{f}_S(s) = \begin{cases} f_X(-A - s) & s > 0 \\ 0 & s = 0 \\ f_X(A - s) & s < 0 \end{cases}$$

$$f_S(s) = \tilde{f}_S(s) + p_3 \delta(s)$$

Saturation Distortion



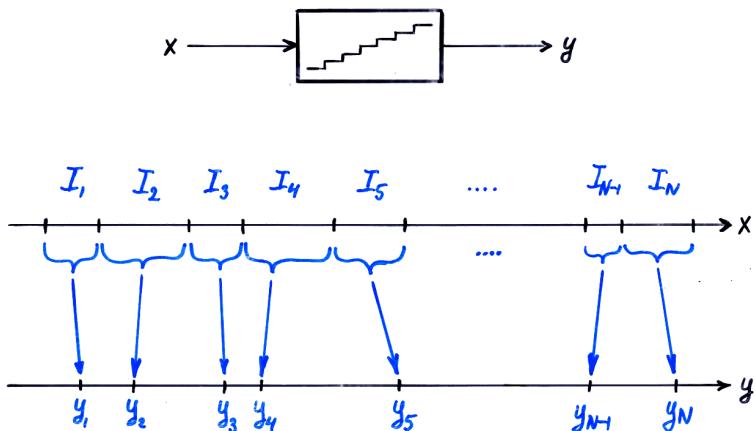
$$\begin{aligned} P_S = E\{S^2\} &= \int_{-\infty}^{\infty} s^2 f_S(s) ds = \int_{-\infty}^0 s^2 f_X(A-s) ds + \int_0^{\infty} s^2 f_X(-A-s) ds \\ &= \int_A^{\infty} (x-A)^2 f_X(x) dx + \int_{-\infty}^{-A} (x+A)^2 f_X(x) dx \end{aligned}$$

SDR – Signal-to-Distortion Ratio

$$SDR = \frac{P_X}{P_S}$$

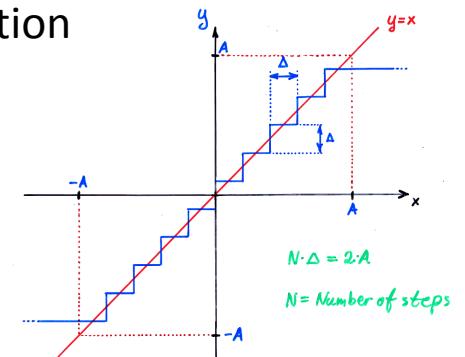
$$SDR_{dB} = 10 \cdot \log_{10} \left(\frac{P_X}{P_S} \right)$$

Quantization Principles

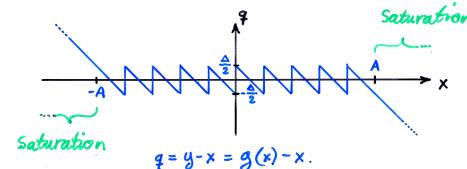


Uniform Quantization

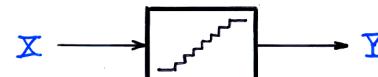
$$y = g(x) = \begin{cases} A - \frac{\Delta}{2}, & x > A \\ \frac{\Delta}{2} + \lfloor \frac{x}{\Delta} \rfloor \cdot \Delta, & |x| \leq A \\ -A + \frac{\Delta}{2}, & x < -A \end{cases}$$



Quantization error:



Quantization Distortion 1(2)



The error: $Q = Y - X = g(X) - X$

Quantization distortion:

$$P_Q = E\{Q^2\} = E\{(g(X) - X)^2\} = \int_{-\infty}^{\infty} (g(x) - x)^2 f_X(x) dx$$

Assumptions:

1. No saturation: $f_X(x) = 0$ for $|x| \geq A$
2. Nice distribution: $f_X(x)$ continuous for $|x| < A$
3. Small Δ : $f_X(x)$ approx. const. in intervals of length Δ .

Quantization Distortion 2(2)

$$\begin{aligned}
 P_Q &= \int_{-A}^A (g(x) - x)^2 f_X(x) dx = \sum_{k=1}^N \int_{y_k-\frac{\Delta}{2}}^{y_k+\frac{\Delta}{2}} (y_k - x)^2 f_X(x) dx = \left| \frac{u=x-y_k}{du=dx} \right| \\
 &= \sum_{k=1}^N \int_{-\Delta/2}^{\Delta/2} u^2 f_X(y_k+u) du \approx \sum_{k=1}^N \int_{-\Delta/2}^{\Delta/2} u^2 f_X(y_k) du \\
 &\quad \text{↑ } \Delta \text{ small } u \text{ small} \\
 &= \sum_{k=1}^N f_X(y_k) \int_{-\Delta/2}^{\Delta/2} u^2 du = \frac{\Delta^2}{12} \sum_{k=1}^N \Delta \cdot f_X(y_k) \approx \frac{\Delta^2}{12} \sum_{k=1}^N \Pr\{X \in I_k\} = \frac{\Delta^2}{12}
 \end{aligned}$$

Error distribution: Approx. uniformly distr. over $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$

Generally without saturation:

$$P_Q \leq \frac{\Delta^2}{4} \text{ since } |Q| \leq \frac{\Delta}{2}.$$

SDR for Uniform Quantization

Still limited to $[-A, A]$ and nice enough distribution.

$$P_Q = E\{Q^2\} = \int_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta} dq = \frac{\Delta^2}{12} = \frac{A^2}{3N^2} \Rightarrow SDR = \frac{P_X}{P_Q} = \frac{3P_X}{A^2} N^2 = \frac{3P_X}{A^2} 2^{2n}$$

$$SDR_{dB} = 10 \log_{10}(SDR) = 10 \log_{10}\left(\frac{3P_X}{A^2}\right) + n \cdot 20 \log_{10}(2) \approx 10 \log_{10}\left(\frac{3P_X}{A^2}\right) + 6n.$$

Example: Uniform distribution over $[-A, A]$.

$$\begin{aligned}
 P_X &= E\{X^2\} = \int_{-A}^A x^2 \frac{1}{2A} dx = \frac{A^2}{3}, \\
 SDR_{dB} &\approx 10 \log_{10}\left(\frac{3A^2/3}{A^2}\right) + 6n = 10 \log_{10}(1) + 6n = 6n
 \end{aligned}$$

SDR for Uniform Quantization with Saturation

Uniform distribution over $[-B, B]$.

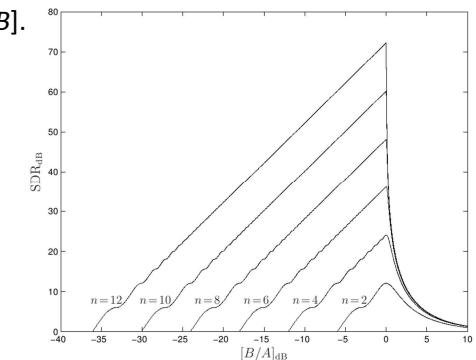
$$P_X = E\{X^2\} = \int_{-B}^B x^2 \frac{1}{2B} dx = \frac{B^2}{3}.$$

Q and S uncorrelated:

$$P_{Q+S} = P_Q + P_S.$$

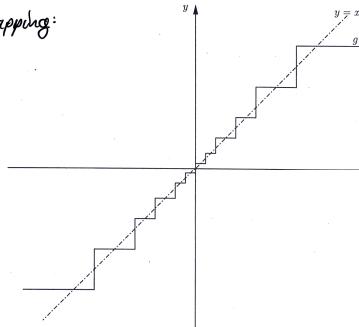
$$SDR = \frac{P_X}{P_Q + P_S},$$

$$P_{Q+S} = \begin{cases} \frac{(2k+1)(\Delta/2)^3 + (B-(2k+1)\Delta/2)^3}{3B}, & k\Delta \leq B < (k+1)\Delta, k \in \{0, 1, \dots, N/2 - 2\} \\ \frac{A-\Delta/2}{B} \cdot \frac{\Delta^2}{12} + \frac{(B-A+\Delta/2)^3}{3B} & B \geq A - \frac{\Delta}{2} \end{cases}$$

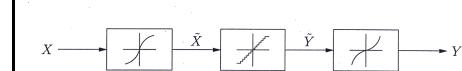


Non-Uniform Quantization

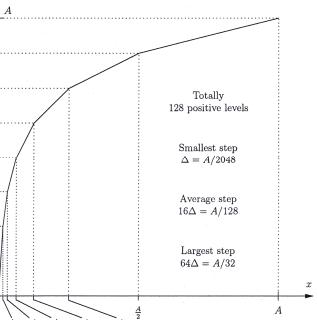
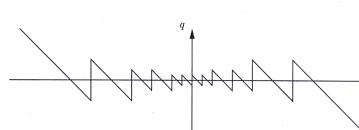
Mapping:



Realization:

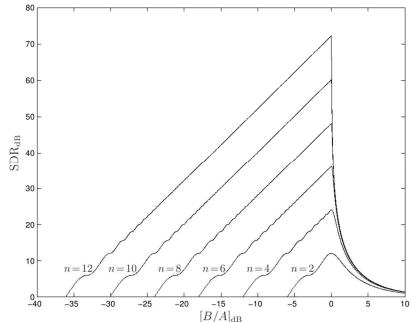


Quantization error:

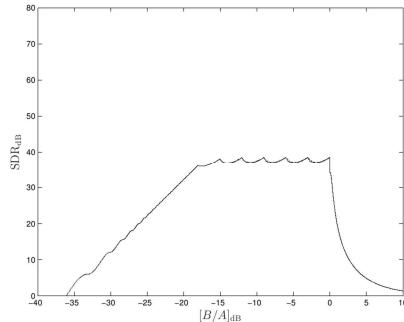


SDR for Non-Uniform Quantization

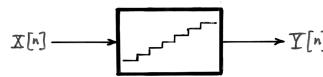
Uniform Quantization



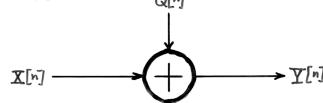
Non-Uniform Quantization



Modelling Quantization of a Stochastic Process



Model:



Model:

- The quantization noise is white
- The input and the quantization noise are uncorrelated.

Reason:

- It is almost true under reasonable assumptions.
- It gets more true with smaller quantization step.

The Quantization Noise is Almost White 1(4)

Quantization Noise: $Q[n] = Y[n] - X[n]$.

Model of PSD: $R_Q[\theta] = \frac{\Delta^2}{12}$

Assumptions:

- that $f_{X[0],X[k]}(x_0, x_k) = 0$ holds for $|x_0| \geq A$ and for $|x_k| \geq A$,
- that $f_{X[0],X[k]}(x_0, x_k)$ is three times differentiable inside the square $|x_0| < A, |x_k| < A$,
- that $f_{X[0],X[k]}(-x_0, -x_k) = f_{X[0],X[k]}(x_0, x_k)$ holds.

Objective:

Show $\frac{r_Q[k]}{r_Q[0]} \rightarrow \delta[k]$, when $\Delta \rightarrow 0$,

The Quantization Noise is Almost White 2(4)

ACF of the Quantization Noise for $k \neq 0$:

$$r_Q[k] = \iint_{-\Delta/2}^{\Delta/2} q_0 q_k f_{Q[0],Q[k]}(q_0, q_k) dq_0 dq_k$$

2-D PDF of the Quantization Noise:

$$\begin{aligned} f_{Q[0],Q[k]}(q_0, q_k) &= \\ &= \begin{cases} \sum_{i_0=1}^{\frac{2A}{\Delta}} \sum_{i_k=1}^{\frac{2A}{\Delta}} f_{X[0],X[k]}(q_0 - A - \frac{\Delta}{2} + i_0 \Delta, q_k - A - \frac{\Delta}{2} + i_k \Delta), & |q_0| < \frac{\Delta}{2}, |q_k| < \frac{\Delta}{2}, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

2-D Taylor series expansion of $f_{X[0],X[k]}(q_0 - A - \frac{\Delta}{2} + i_0 \Delta, q_k - A - \frac{\Delta}{2} + i_k \Delta)$ gives us:

$$f_{Q[0],Q[k]}(q_0, q_k) = \frac{K_1 + K_2 q_0 + K_3 q_k + K_4 q_0^2 + K_5 q_0 q_k + K_6 q_k^2 + K_7 (q_0^2 + q_k^2)^{3/2}}{\Delta^2}$$

The Quantization Noise is Almost White 3(4)

What about all those coefficients? Symmetry:

$$f_{X[0],X[k]}(-x_0, -x_k) = f_{X[0],X[k]}(x_0, x_k) \Rightarrow K_2 = K_3 = 0$$

Result:

$$r_Q[k] = \iint_{-\Delta/2}^{\Delta/2} q_0 q_k \frac{K_1 + K_4 q_0^2 + K_5 q_0 q_k + K_6 q_k^2 + K_7 (q_0^2 + q_k^2)^{3/2}}{\Delta^2} dq_0 dq_k.$$

Observation:

$$\iint_{-\Delta/2}^{\Delta/2} q_0 q_k dq_0 dq_k = \iint_{-\Delta/2}^{\Delta/2} q_0^3 q_k dq_0 dq_k = \iint_{-\Delta/2}^{\Delta/2} q_0 q_k^3 dq_0 dq_k = 0$$

Result:

$$r_Q[k] = \iint_{-\Delta/2}^{\Delta/2} \frac{K_5 q_0^2 q_k^2 + K_7 q_0 q_k (q_0^2 + q_k^2)^{3/2}}{\Delta^2} dq_0 dq_k.$$

The Quantization Noise is Almost White 4(4)

We had:

$$r_Q[k] = \iint_{-\Delta/2}^{\Delta/2} \frac{K_5 q_0^2 q_k^2 + K_7 q_0 q_k (q_0^2 + q_k^2)^{3/2}}{\Delta^2} dq_0 dq_k.$$

Upper bound:

$$|r_Q[k]| < K \Delta^4 \Rightarrow \frac{|r_Q[k]|}{r_Q[0]} < 12K \Delta^2.$$

Result:

$$\frac{r_Q[k]}{r_Q[0]} \rightarrow \delta[k], \text{ when } \Delta \rightarrow 0.$$

Conclusion:

Almost white. Closer to white as Δ decreases.

Input & Quantization Noise Almost Uncorrelated

Same assumptions:

1. that $f_{X[0],X[k]}(x_0, x_k) = 0$ holds for $|x_0| \geq A$ and for $|x_k| \geq A$,
2. that $f_{X[0],X[k]}(x_0, x_k)$ is three times differentiable inside the square $|x_0| < A, |x_k| < A$,
3. that $f_{X[0],X[k]}(-x_0, -x_k) = f_{X[0],X[k]}(x_0, x_k)$ holds.

Normalized cross-covariance:

$$\rho_{XQ}[k] = \frac{\text{Cov}\{X[0], Q[k]\}}{\sigma_X \sigma_Q} = \frac{\mathbb{E}\{(X[0] - m_X)(Q[k] - m_Q)\}}{\sigma_X \sigma_Q}$$

Objective:

Show $\rho_{XQ}[k] \rightarrow 0$, when $\Delta \rightarrow 0$, for all k .

Modelling Quantization Noise

Observation:

$$\begin{aligned} f_{X[0],X[k]}(-x_0, -x_k) &= f_{X[0],X[k]}(x_0, x_k) \\ \Rightarrow f_X(x) \& \text{ & } f_Q(q) \text{ even } \Rightarrow m_Q = m_X = 0 \\ \Rightarrow \rho_{XQ}[k] &= \frac{r_{XQ}[k]}{\sigma_X \sigma_Q} = \frac{\mathbb{E}\{X[0]Q[k]\}}{\sigma_X \sigma_Q} \end{aligned}$$

Similar reasoning as before:

$$|\rho_{XQ}[k]| < \sqrt{12}K\Delta/\sigma_X \Rightarrow \rho_{XQ}[k] \rightarrow 0, \text{ when } \Delta \rightarrow 0, \text{ for all } k.$$

Conclusion:

Almost uncorrelated. Less correlated as Δ decreases.

Quantization – ACF & PSD Relations

Assumptions:

- $Q[n]$ is uniformly distributed on $[-\Delta/2, \Delta/2]$.
- $Q[n]$ is a white process.
- $Q[n]$ and $X[n]$ are uncorrelated.

ACF of output:

$$r_Y[k] = E\{Y[n]Y[n+k]\} = E\{(X[n]+Q[n])(X[n+k]+Q[n+k])\}$$

$$= E\{X[n]X[n+k]\} + E\{X[n]Q[n+k]\} + E\{Q[n]X[n+k]\} + E\{Q[n]Q[n+k]\}.$$

Uncorrelated processes:

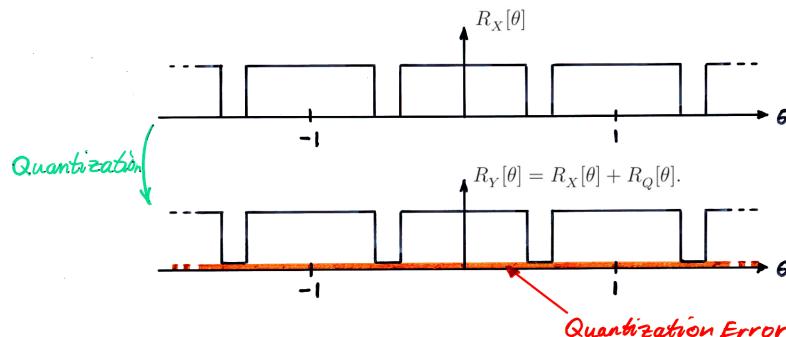
$$E\{X[n]Q[n+k]\} = E\{Q[n]X[n+k]\} = m_X m_Q = 0,$$

Result:

$$r_Y[k] = r_X[k] + r_Q[k].$$

$$R_Y[\theta] = R_X[\theta] + R_Q[\theta].$$

Quantization – Power-Spectral Densities



Poisson Processes 1(2)

A counting process, $\mathbb{X}(t)$. Time continuous & amplitude discrete.
Counting the number of arrivals so far.

Examples:

Customers entering a shop.

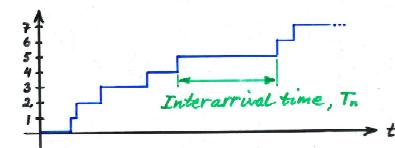
Cars passing by

Radio-active decay

Light (counting photons)

Packets in a network

A realization:



Properties of a stationary Poisson process:

$$\mathbb{X}(0) = 0$$

$$\mathbb{X}(t_1) \leq \mathbb{X}(t_2) \text{ if } t_1 \leq t_2.$$

$$\Pr\{\mathbb{X}(t)=k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \geq 0, k \in \mathbb{N}$$

$$\Pr\{\mathbb{X}(t+\tau)-\mathbb{X}(t)=k\} = \frac{(\lambda \tau)^k}{k!} e^{-\lambda \tau}, \quad t \geq 0 \text{ & } \tau \geq 0, k \in \mathbb{N}$$

Interarrival times:

T_m & T_n indep. for $m \neq n$

$$f_{T_n}(t_n) = \lambda e^{-\lambda t_n}, \quad t_n \geq 0$$

$$F_{T_n}(t_n) = 1 - e^{-\lambda t_n}, \quad t_n \geq 0$$

Poisson Processes 2(2)

Expectation: $E\{\mathbb{X}(t)\} = \sum_{k=0}^{\infty} k \Pr\{\mathbb{X}(t)=k\} = \lambda t$ (λ intensity)

Variance: $\text{Var}\{\mathbb{X}(t)\} = \lambda t$ for $t \geq 0$

Power: $E\{\mathbb{X}^2(t)\} = E^2\{\mathbb{X}(t)\} + \text{Var}\{\mathbb{X}(t)\} = \lambda t(1+\lambda t)$

Increments: $\mathbb{X}(t+\tau) - \mathbb{X}(t)$. Nonoverlapping increments are independent.

ACF ($0 \leq t_1 \leq t_2$): $r_{\mathbb{X}}(t_1, t_2) = E\{\mathbb{X}(t_1)\mathbb{X}(t_2)\} = E\{\mathbb{X}(t_1)(\mathbb{X}(t_2) - \mathbb{X}(t_1) + \mathbb{X}(t_1))\}$

$$= E\{(\mathbb{X}(t_1) - \mathbb{X}(0))(\mathbb{X}(t_2) - \mathbb{X}(t_1))\} + E\{\mathbb{X}^2(t_1)\}$$

Indep. incr, $0 \leq t_1 \leq t_2$

$$= E\{\mathbb{X}(t_1) - \mathbb{X}(0)\} \cdot E\{\mathbb{X}(t_2) - \mathbb{X}(t_1)\} + E\{\mathbb{X}^2(t_1)\}$$

$$= \lambda t_1 (\lambda t_2 - \lambda t_1) + \lambda t_1 (1 + \lambda t_1) = \lambda t_1 (1 + \lambda t_2)$$

ACF ($0 \leq t_2 \leq t_1$): $r_{\mathbb{X}}(t_1, t_2) = \lambda t_2 (1 + \lambda t_1)$ (similarly)

ACF (total): $r_{\mathbb{X}}(t_1, t_2) = \lambda \cdot \min\{t_1, t_2\} + \lambda^2 t_1 t_2$

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