TSDT14 Signal Theory

Solutions to the exam 2019-08-28

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1

These sub-tasks have to be treated correctly as partial fulfillment to pass the exam:

a. The most simple way to define a time-continuous stochastic process is to describe it as a stochastic variable, whose realizations are time-continuous signals.

b. The output $Y[n]$ has PSD

$$R_Y[\theta] = \left|H[\theta]\right|^2 R_X[\theta] = 4R_0 \sin^2(\pi\theta).$$

c. The output $Y[n]$ has ACF

$$r_Y[k] = 2r_X^2[k] + r_X^2[0] = 2\left(\frac{1}{9}\right)^{|k|} + 1$$

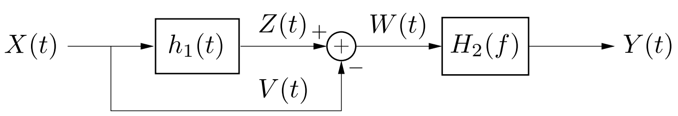
according to the tables and formulas, since the process is Gaussian with mean zero.

2

Let $X(t)$ in the figure below be given by

$$X(t) = A + B(t),$$

where A is a Gaussian variable with mean $m_A = 1$ and variance $\sigma_A^2 = 2$, and where $B(t)$ is a Gaussian WSS process with mean zero and ACF $r_B(\tau) = R_0\,\mathrm{sinc}(20\tau)$.



In this figure, we have introduced the new processes $Z(t)$, $V(t)$ and $W(t)$. The PSD of $B(t)$ is given by

$$R_B(f) = \mathcal{F}\{r_B(\tau)\} = \frac{R_0}{20}\,\mathrm{rect}(\tau/20).$$

A is a stochastic varable. Viewed as a process $A(t)$ it has constant realizations, and its ACF is given by

$$r_A(f) = \mathrm{E}\{A^2\} = m_A^2 + \sigma_A^2 = 3,$$

and its PSD is

$$R_A(f) = \mathcal{F}\{r_A(\tau)\} = 3\delta(f).$$

A and $B(f)$ are independent. Thus the PSD of $X(t)$ is given by

$$R_X(f) = R_A(f) + R_B(f) = 3\delta(f) + \frac{R_0}{20}\,\mathrm{rect}(f/20).$$

Its power is

$$P_X = \int_{-\infty}^{\infty} R_X(f)\,df = \int_{-\infty}^{\infty} R_A(f)\,df + \int_{-\infty}^{\infty} R_B(f)\,df$$
$$= 3 + R_0.$$

The first filter has impulse response

$$h_1(t) = \delta(t) + \delta(t-1),$$

where $\delta(t)$ as is the time-continuous unit impulse. Thus, the output of that filter is given by

$$Z(t) = X(t) + X(t-1).$$

The signal $V(t)$ is obviously given by

$$V(t) = Z(t) - X(t) = X(t-1).$$

So, we have

$$R_V(f) = R_X(f).$$

The second filter has frequency response

$$H_2(f) = \mathrm{rect}(f/10).$$

Thus, the output $Y(t)$ has PSD

$$R_Y(f) = \left|H_2(f)\right|^2 R_X(f) = 3\delta(f) + \frac{R_0}{20}\,\mathrm{rect}(f/10),$$

and its power is

$$P_Y = \int_{-\infty}^{\infty} R_Y(f)\,df = 3 + R_0/2.$$

We want to determine R_0 such that we have

$$P_Y = \frac{2}{3} P_X.$$

This gives us the equation

$$3 + \frac{R_0}{2} = \frac{2}{3}(3 + R_0),$$

which has the solution $R_0 = 6$.

3
A quantization task...

a. With a small quantization step Δ and a nice enough probability density function, we can assume that the quantization noise $Q[n]$ is uniformly distributed over $[-\Delta/2, \Delta/2)$. This gives us

$$\mathrm{E}\{Q^2[n]\} = \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta} \cdot q^2 \, dq = \frac{\Delta^2}{12}.$$

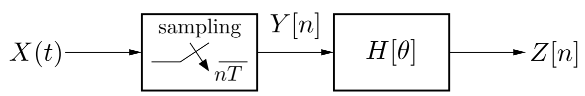
b. If $Q[n]$ is uniformly distributed over $[-\Delta/2, \Delta/2)$, we have trivially

$$\Pr\{|Q[n]| > \Delta/2\} = 0.$$

c. The sequence $X[n]$ consists of independent symbols. Then that also holds for $Q[n]$. But $Q[n]$ has mean zero and the power $\Delta^2/12$ according to part a. Then we have the PSD

$$R_Q[\theta] = \frac{\Delta^2}{12}.$$

4
We have the following situation.



The input $X(t)$ has PSD

$$R_X(f) = \begin{cases} R_0, & |f| < B, \\ 0, & \text{elsewhere} \end{cases}$$

and it is sampled with sampling frequency $f_s = 4B$. The bandwidth of the signal is obviously less than half the sampling frequency, and thus there is no aliasing in the sampling. The sampled signal has PSD

$$R_Y[\theta] = f_s \sum_k R_X\big(f_s(\theta - k)\big) = \begin{cases} 4BR_0, & |\theta| < 1/4, \\ 0, & 1/4 \le |\theta| \le 1/2, \\ R_Y[\theta + m] & m \text{ integer.} \end{cases}$$

For the frequency response of the filter, we have

$$|H[\theta]|^2 = \begin{cases} 1 - 2|\theta|, & |\theta| < \frac{1}{2}, \\ |H[\theta] + m|^2, & m \text{ integer.} \end{cases}$$

Its output has PSD

$$R_Z[\theta] = |H[\theta]|^2 R_Y[\theta] = \begin{cases} 4BR_0 \cdot (1 - 2|\theta|), & |\theta| < 1/4, \\ 0, & 1/4 \le |\theta| \le 1/2, \\ R_Z[\theta + m] & m \text{ integer.} \end{cases}$$

Finally, we determine the signal power (the quadratic mean) as

$$P_Z = \mathrm{E}\{Z^2[n]\} = r_Z[0] = \int_{-1/2}^{1/2} R_Z[\theta] \, d\theta = \int_{-1/4}^{1/4} 4BR_0 \cdot (1 - 2|\theta|) \, d\theta = \frac{3R_0B}{2}$$

5
Let $R(f)$ be the Fourier Transform of $r(\tau)$. If $r(\tau)$ is the ACF of a WSS process, then $R(f)$ is its PSD, and vice versa. Necessary conditions of $r(\tau)$ to be the ACF of a WSS process is that it is real-valued, even, and has it largest absolute value in the origin. Sufficient condition for $R(f)$ to be the PSD of a WSS process is that it is real-valued and non-negative.

a. We have

$$r(\tau) = \mathrm{rect}(\tau) \quad \Leftrightarrow \quad R(f) = \mathrm{sinc}(f).$$

Obviously, $R(f)$ is negative for some values of f. Thus, $R(f)$ is not a PSD and consequently, $r(\tau)$ is not an ACF of a WSS process.

b. We have

$$r(\tau) = \mathrm{triangle}(\tau) \quad \Leftrightarrow \quad R(f) = \mathrm{sinc}^2(f).$$

Obviously, $R(f)$ is real-valued and non-negative for all f. Thus, $R(f)$ is a PSD and consequently, $r(\tau)$ is an ACF of a WSS process.

c. We have

$$r(\tau) = \text{rect}(\tau) + \text{triangle}(\tau)$$
$$\Leftrightarrow$$
$$R(f) = \text{sinc}(f)\big(1 + \text{sinc}(f)\big).$$

We notice that the expression in parentheses is positive for all non-zero f. Multiplied with that is $\text{sinc}(f)$, which is negative for some non-zero f. As a result, $R(f)$ is negative for some values of f. Thus, $R(f)$ is not a PSD and consequently, $r(\tau)$ is not an ACF of a WSS process.

d. We have

$$r(\tau) = \delta(\tau + 1).$$

This is not an even function and consequently, $r(\tau)$ is not an ACF of a WSS process.

e. We have

$$r(\tau) = 3\,\text{triangle}(\tau) - 2\,\text{triangle}(\tau/2)$$
$$\Leftrightarrow$$
$$R(f) = 3\,\text{sinc}^2(f) - 4\,\text{sinc}^2(2f).$$

We see that $R(0)$ is negative. Thus, $R(f)$ is not a PSD and consequently, $r(\tau)$ is not an ACF of a WSS process.

6

We are given the ACF $r_X(\tau) = \text{sinc}(10\tau)$. The PSD of $X(t)$ is then the Fourier transform of that, which according to Page 19 in T&F is

$$R_X(f) = 0.1 \cdot \text{rect}(f/10).$$

Then we have the modulated signal

$$Y(t) = 10X(t)\cos(2000\pi t + \Psi),$$

where Ψ is uniformly distributed over $[0, 2\pi)$ and independent of $X(t)$. The corresponding PSD is according to T&F, Page 14, given by

$$R_Y(f) = 25\big(R_X(f + f_c) + R_X(f - f_c)\big),$$

where f_c is the carrier frequency 1000 Hz. This gives us

$$R_Y(f) = \begin{cases} 2.5, & 995 < |f| < 1005, \\ 0, & \text{elsewhere.} \end{cases}$$

This signal is then demodulated coherently as

$$Z(t) = 2Y(t)\cos(2000\pi t + \Psi),$$

where we notice particularily that the random phase Ψ is *the same* random phase as for the modulation. This can be written as

$$\begin{aligned} Z(t) &= 20X(t)\cos^2(2000\pi t + \Psi) \\ &= 10X(t)\big(1 + \cos(4000\pi t + 2\Psi)\big). \end{aligned}$$

This is thus the sum of a scaled version of the original signal and an amplitude modulated version, modulated with carrier frequency $2f_c = 2000$ Hz. We notice that the two parts do not overlap in the frequency domain. The resulting PSD is then

$$\begin{aligned} R_Z(f) &= 100R_X(f) + 25\big(R_X(f + 2f_c) + R_X(f - 2f_c)\big) \\ &= \begin{cases} 10, & |f| < 5, \\ 2.5, & 1995 < |f| < 2005, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

This signal is filtered by the ideal LP filter with frequency response

$$H(f) = \begin{cases} 1, & |f| < 20, \\ 0, & \text{elsewhere.} \end{cases}$$

Its output $W(t)$ has PSD

$$\begin{aligned} R_W(f) &= \big|H(f)\big|^2 R_Y(f) \\ &= \begin{cases} 10, & |f| < 5, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

Finally, all those PSD as graphs: