

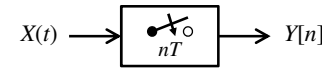
TSDT14 Signal Theory

Lecture 9

Reconstruction and Reconstruction Errors

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Sampling and PAM of WSS Processes – Summary

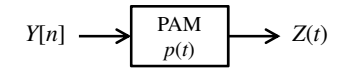


$$Y[n] = X(nT)$$

$$m_Y = m_X$$

$$r_Y[k] = r_X(kT)$$

$$R_Y[\theta] = \frac{1}{T} \sum_m R_X\left(\frac{\theta - m}{T}\right)$$



$$Z(t) = \sum_n Y[n] p(t - nT - \Psi)$$

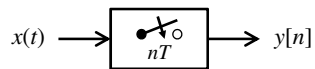
$$m_Z = \frac{1}{T} P(0) m_Y$$

Ψ unif. $[0, T]$
 Ψ & $Y[n]$ indep.

$$r_Z(\tau) = \frac{1}{T} \sum_k r_Y[k] (p * \tilde{p})(\tau - kT)$$

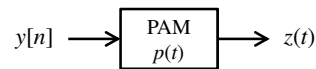
$$R_Z(f) = \frac{1}{T} |P(f)|^2 R_Y[fT]$$

Sampling and PAM of Deterministic Signals – Summary



$$y[n] = x(nT)$$

$$Y[\theta] = \frac{1}{T} \sum_m X\left(\frac{\theta - m}{T}\right)$$

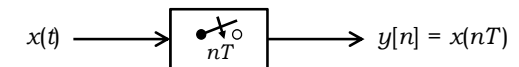


$$z(t) = \sum_n y[n] p(t - nT)$$

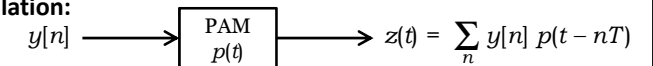
$$Z(f) = P(f) Y[fT]$$

Linear Mappings

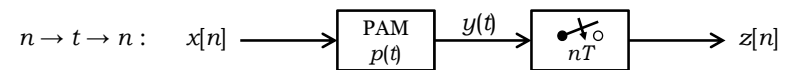
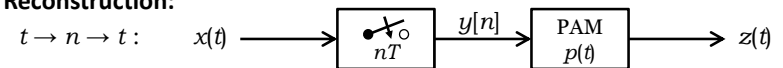
Sampling:



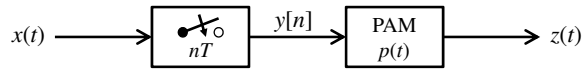
Pulse-Amplitude Modulation: (PAM)



Reconstruction:



Sampling Theorem for Deterministic Signals



The Sampling Theorem:

Consider a signal $x(t)$, with spectrum $X(f)$ and $X(f) = 0$ for $|f| \geq f_0$. If $x(t)$ is sampled with sampling frequency f_s , then $x(t)$ can be reconstructed without error from the sampled signal if $f_s \geq 2f_0$ holds.

This means:

There exists a pulse shape $p(t)$, such that $x(t)$ can be written as

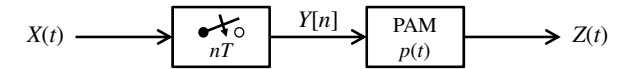
$$x(t) = \sum_n x(nT) p(t - nT)$$

if $f_s \geq 2f_0$ holds, where $f_s = 1/T$.

Fulfilled for:

Ideal reconstruction: $p(t) = \text{sinc}(t/T)$

The Sampling Theorem for Stochastic Processes



The Sampling Theorem:

Consider a process $X(t)$, with spectrum $R_X(f)$ and $R_X(f) = 0$ for $|f| \geq f_0$. If $X(t)$ is sampled with sampling frequency f_s , then $X(t)$ can be reconstructed without error from the sampled signal if $f_s \geq 2f_0$ holds.

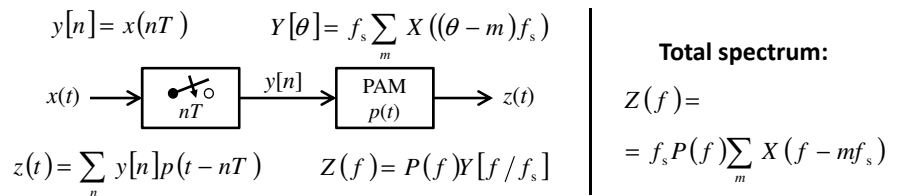
This means:

There exists a pulse shape $p(t)$, such that $\varepsilon^2 = E\{(Z(t) - X(t))^2\} = 0$ holds.

Fulfilled for:

Ideal reconstruction: $p(t) = \text{sinc}(t/T)$

Reconstruction – Deterministic Case $t \rightarrow n \rightarrow t$:



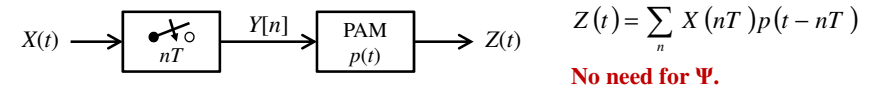
Distorsion: $\varepsilon^2 = \int_{-\infty}^{\infty} (z(t) - x(t))^2 dt = \int_{-\infty}^{\infty} |Z(f) - X(f)|^2 df$

Parseval
 $= \int_{-\infty}^{\infty} \left| f_s P(f) \sum_m X(f - mf_s) - X(f) \right|^2 df = \int_{-\infty}^{\infty} \left| (f_s P(f) - 1) X(f) + f_s P(f) \sum_{m \neq 0} X(f - mf_s) \right|^2 df$

Ideal reconstruction: $\varepsilon^2 = 2 \int_{f_s/2}^{\infty} |X(f)|^2 df + 2 \int_0^{f_s/2} \left| \sum_{m \neq 0} X(f - mf_s) \right|^2 df$

$p(t) = \text{sinc}(t/T)$ **Bandlimiting distortion** **Aliasing distortion**
 $P(f) = T \text{rect}(fT)$

Proof of Sampling Theorem 1(2)



Distorsion: $\varepsilon^2 = E\{(Z(t) - X(t))^2\} = E\{Z^2(t)\} - 2E\{Z(t)X(t)\} + E\{X^2(t)\}$

$E\{Z^2(t)\} = E\left\{ \left(\sum_n X(nT) p(t - nT) \right)^2 \right\} = \sum_m \sum_n E\{X(nT)X(mT)\} p(t - nT) p(t - mT)$

$= \sum_m \left(\sum_n r_x(nT - mT) p(t - nT) \right) p(t - mT)$

$E\{Z(t)X(t)\} = E\left\{ \sum_n X(nT) p(t - nT) X(t) \right\} = \sum_n E\{X(nT)X(t)\} p(t - nT)$

$= \sum_n r_x(nT - t) p(t - nT)$

$E\{X^2(t)\} = r_x(0)$

Proof of Sampling Theorem 2(2)

We had:

$$E\{Z^2(t)\} = \sum_m \left(\sum_n r_x(nT - mT) p(t - nT) \right) p(t - mT)$$

$$E\{Z(t)X(t)\} = \sum_n r_x(nT - t) p(t - nT)$$

Try ideal reconstruction:

$$p(t) = \text{sinc}(t/T)$$

From the deterministic case: $r_x(\tau) = \sum_m r_x(mT) p(\tau - mT)$

$$r_x(\tau - a) = \sum_n r_x(nT) p(\tau - a - nT) = \sum_n r_x(nT - a) p(\tau - nT) \quad (1)$$

With $\tau = t$ & $a = mT$ in (1), we get: $r_x(t - mT) = \sum_n r_x(nT - mT) p(t - nT)$ (2)

With $\tau = a = t$ in (1), we get: $r_x(0) = \sum_n r_x(nT - t) p(t - nT)$ (3)

$$(2) \& (3) \Rightarrow E\{Z^2(t)\} = \sum_m r_x(t - mT) p(t - mT) = \sum_m r_x(mT - t) p(t - mT) = r_x(0)$$

$$(3) \Rightarrow E\{Z(t)X(t)\} = r_x(0) \quad \text{Result: } \epsilon^2 = 0$$

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