TSKS01 Digital Communication
Lecture 1

Introduction, Repetition, Channels as Filters,
Complex-baseband representation

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COMMUNICATION SYSTEMS

LINKÖPING UNIVERSITY

Emil Björnson

- Course Director
  - TSKS01 Digital Communication
  - TSKS12 Multiple Antenna Communication
- Associate Professor (Biträdande professor)
  - Docent in Communication Systems, PhD in Telecommunications
  - Coordinator of Master programme in Communication Systems and Master profile in Communication Systems
  - Researcher on 5G communications (e.g., multiple antenna communications). Theoretical research and inventor.

TSKS01 Digital Communication - Formalities

Information:	www.commsys.isy.liu.se/TSKS01 and LISAM
Lecturer & examiner:	Emil Björnson, emil.bjornson@liu.se
Tutorials:	Kamil Senel, kamil.senel@liu.se
Teaching activities:	12 lectures, 12 tutorials, lab exercises

Examination:

Laboratory exercises (1 hp):
New lab: “deep learning vs. digital communication”
More details in HT2

Written exam (5 hp):

1 simple task (basics)	Min: 50%
2 questions (5 points each)	Min: 3p
4 problems (5 points each)	Min: 6p

Pass: 14 points from questions & problems

Course Aims

After passing the course, the student should

- be able to reliably perform standard calculations regarding digital modulation and binary (linear) codes for error control coding.

  (basics)

- should be able, to some extent, to perform calculations for solutions to practical engineering problems that arise in communication

  (primarily questions)

- be able to, with some precision, analyze and compare various choices of digital modulation methods and coding methods in terms of error probabilities, minimum distances, throughput, and related concepts.

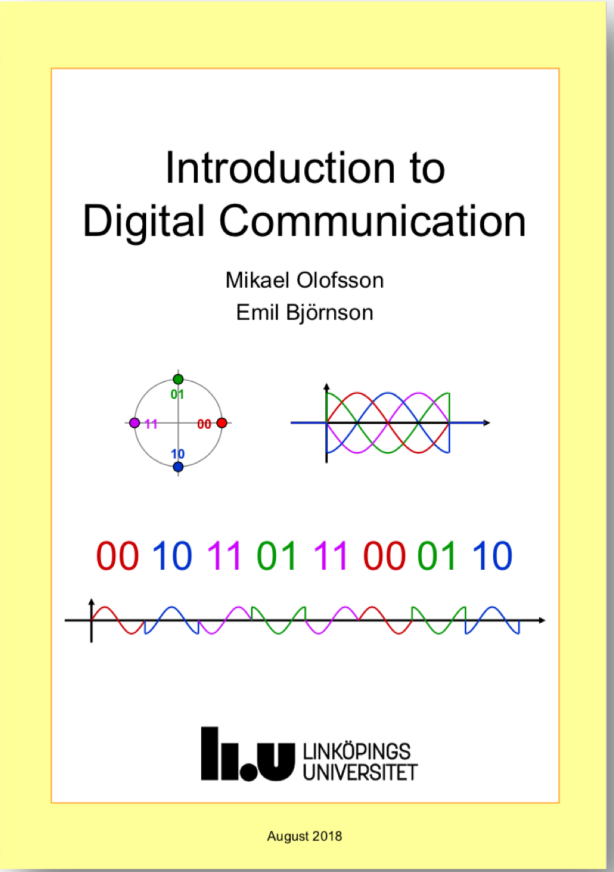
  (problems)

- be able, to some extent, to implement and evaluate communication systems of the kinds treated in the course.

  (laboratory exercises)

Course Book

- Introduction to Digital Communication
  - New edition for 2018, A1758, SEK 272
    - Contains theory and tutorial problems
    - 274 pages dedicated to this course
  - Available for purchase in Building A, LiU Service Center, entrance 19C



- Older editions
  - Version from 2017 (two books) can be used, but is not recommended. Follow last year's course website for list of tutorial problems.

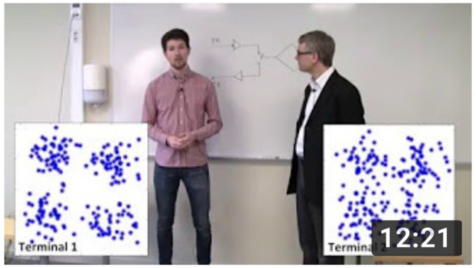
Outline

- Introduction, lecture plan
- Briefly: Signals and systems, Fourier transforms, probability theory
- Complex baseband modeling
- Basic communication example
- Stochastic processes – important for noise modeling

Popular science videos on Youtube:

Communication Systems, Linköping University, LIU

https://www.youtube.com/channel/UCOrjRoYJPqGiR1SZvU3xcYQ/featured



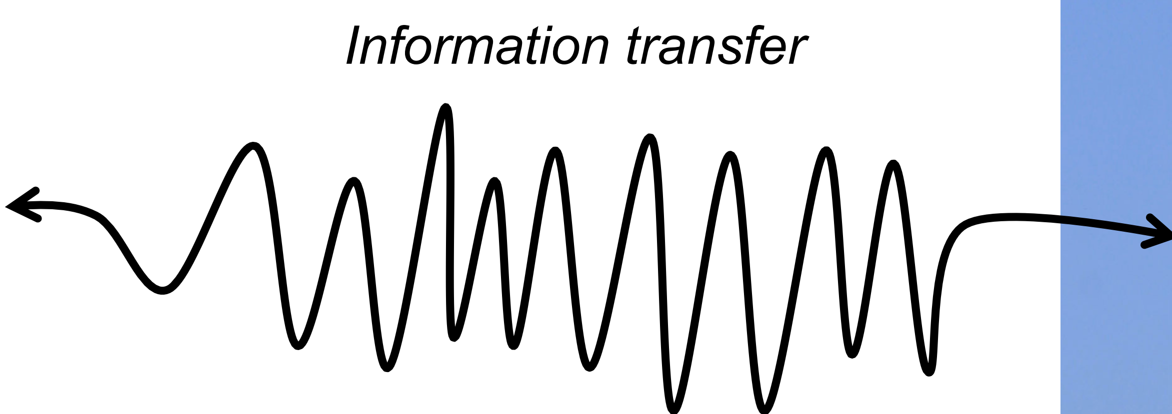
Acoustic Massive MIMO Testbed

Communication Systems, Link...
2.8K views • 1 year ago

Massive Audio Beamforming (TSKS05 Project, 2016)

Communication Systems, Link...
999 views • 8 months ago

What is Communication?



Classic technology

- Newspapers, books
- Telegraph
- Radio and TV broadcast

Modern digital technology

- CD, DVD, Blu-ray
- Optical fibres, network cables
- Wireless (4G, WiFi)

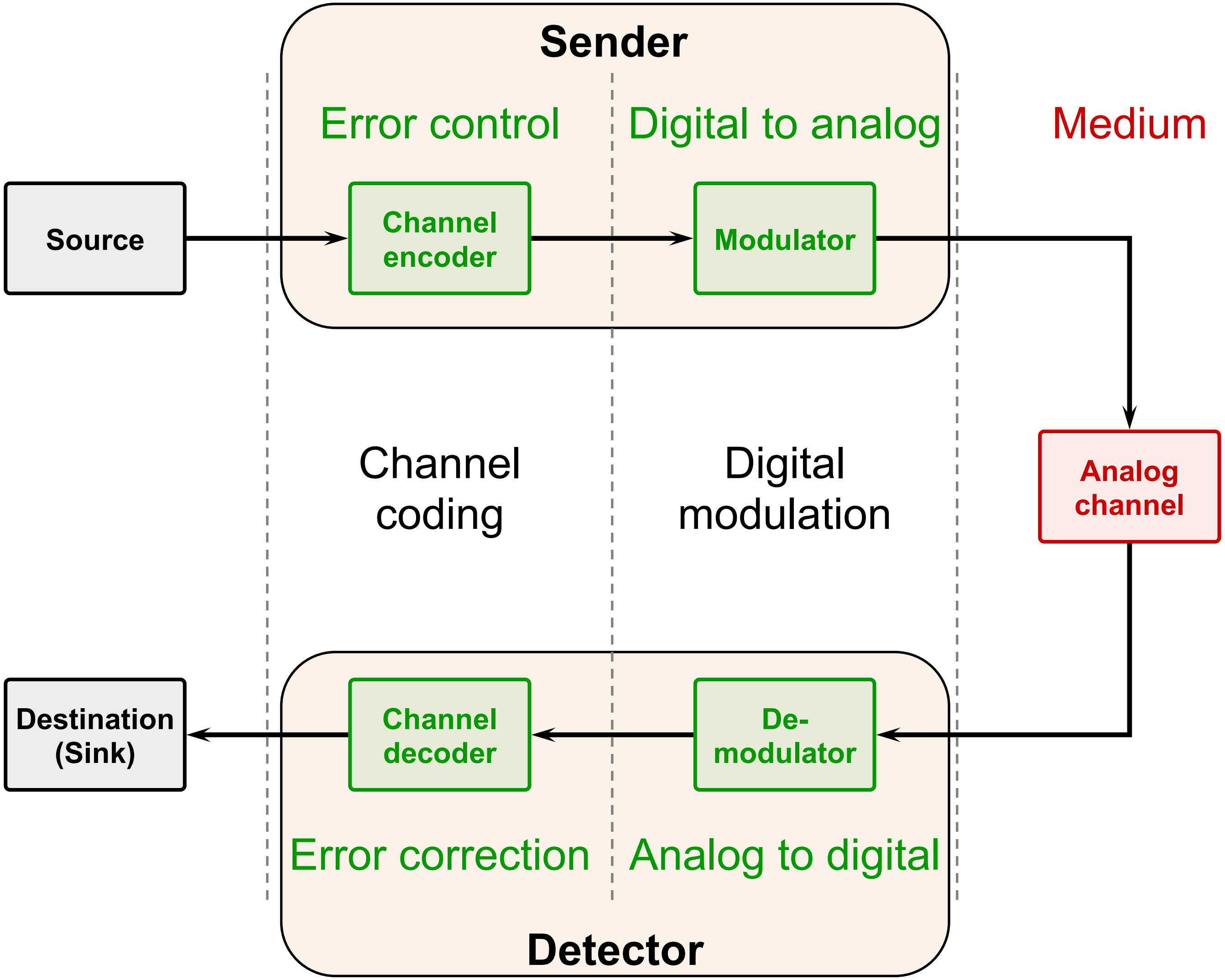
Why **Digital** Communication?

1. **Efficient Source Description**
   - Information has a built-in redundancy (e.g., text, images, sound)
   - Compression: Represent information with minimal number of bits
   - Sequence of bits: 0 and 1
   - 100 bits represent $2^{100} \approx 1.2677 \cdot 10^{30}$ different messages

2. **Efficient Transmission**
   - Nyquist-Shannon Sampling Theorem: B Hz signal $\leftrightarrow$ $2B$ samples per s
   - Sequence of bits is used to select these samples

3. **Guaranteed Quality**
   - Protect signals against errors: All or nothing is received

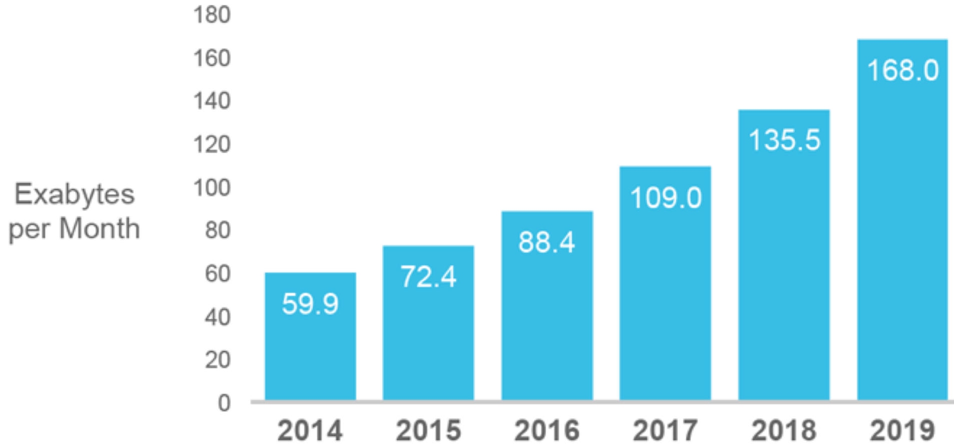
One-way Digital Communication System



Digital Communication: An Exponential Story

23% CAGR 2014–2019

Overall IP traffic



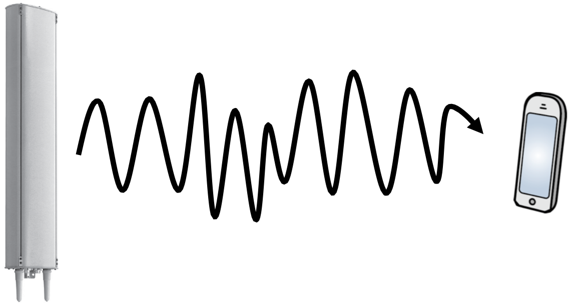
Source: Cisco VNI Global IP Traffic Forecast, 2014–2019

Digital Communication Devices are Everywhere

- PCs, tablets, phones, machine-to-machine, TVs, etc.
- Total internet traffic: 23% growth/year
- Number of devices: 12% growth/year

How can we sustain this exponential growth?

Information-Bearing Signals



- Example: Mobile communication
  - Analog electromagnetic signaling
  - Bandwidth B Hz
  - Sample principle in copper cables, optical fibers, etc.

Nyquist–Shannon Sampling Theorem

- Signal is determined by $2B$ real samples/second (or B complex samples)

- Information-bearing signals
  - Select the samples in a particular way → Represent digital information
  - Spectral efficiency: Number of bits transferred per sample [bit/s/Hz]

Designing Efficient Communication Systems

- Performance metric: Bit rate [bit/s]

$$\text{Bit rate [bit/s]} = \text{Bandwidth [Hz]} \cdot \text{Spectral efficiency [bit/s/Hz]}$$

- How do we increase the bit rate?
  1. Use more bandwidth
  2. Increase spectral efficiency

Wired: Use cable that handles wider bandwidths
Wireless: Buy more spectrum

Each sample represents more bits
- More sensitive to disturbances
- Improve signal by smart modulation and demodulation
- Protect against bit errors with coding

This course deals with spectral efficiency!

Preliminary Lecture Plan

HT1:

- Introduction, repetition, noise modeling ***(Lecture 1)***
- Basic digital modulation ***(Lectures 2-3)***
- Detection in AWGN channels, modulation schemes ***(Lectures 3-5)***
- Detection in dispersive channels ***(Lectures 6-7)***

HT2:

- Error control coding ***(Lectures 8-10)***
- Practical aspects (e.g., synchronization), lab intro ***(Lecture 11)***
- Link adaptation ***(Lecture 12)***

What You are Expected to Know

- Signals and systems
  - Typical signals, LTI systems, impulse response
- Fourier transform
  - Applied to signals and systems
- Probability theory
  - Stochastic variables, mean value, variance, etc.

Repetition videos in Lisam

Repetition: Signals and Systems

Signals: Voltages, currents, or other measurements

Systems: Manipulate/filter signals

$$x(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y(t)$$

Complex exponential: $e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$

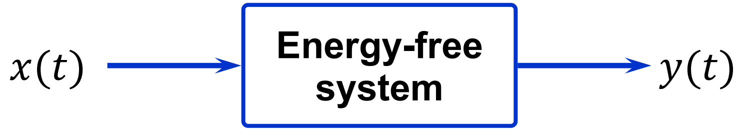
Unit step: $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$

Unit impulse: $\delta(t)$: $\int_{-\infty}^{\infty} x(t)\,\delta(t-a)\,dt = x(a)$

Property: $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

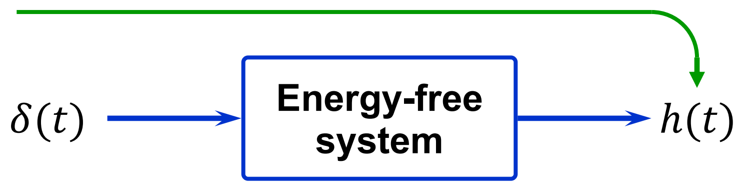
Properties of systems

General case:



Energy-free system: No transients, constant input → constant output

Impulse response:



Linear time-invariant (LTI) system

- Linear: Output is scaled, time-delayed versions of input
- Time-invariant: Always reacts in the same way

Convolution and Output of System

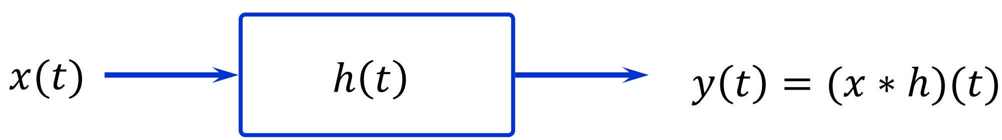
Definition: The convolution of the signals $a(t)$ and $b(t)$ is denoted by $(a * b)(t)$ and defined as

$$(a * b)(t) = \int_{-\infty}^{\infty} a(\tau)b(t-\tau)d\tau\,.$$

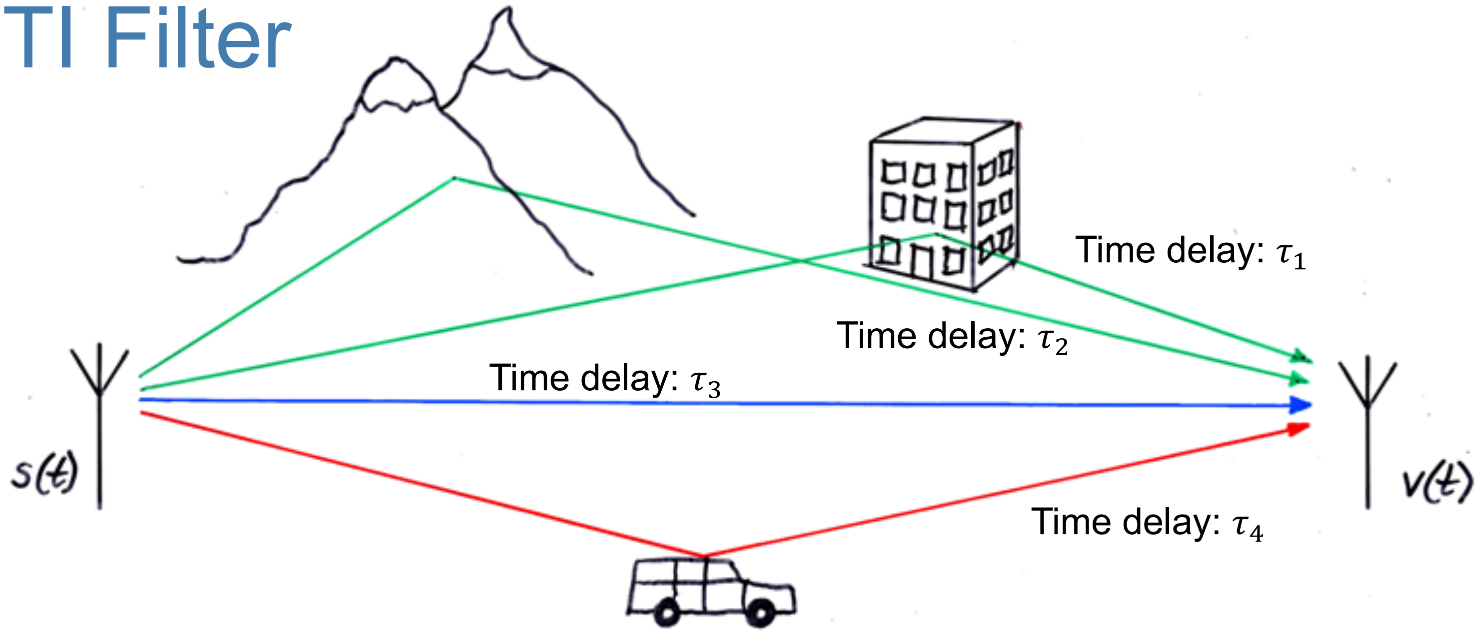
Commutative operation: $(a * b)(t) = (b * a)(t)$.

Theorem: Let $x(t)$ be the input to an energy-free LTI system with impulse response $h(t)$, then the output of the system is

$$y(t) = (x * h)(t).$$



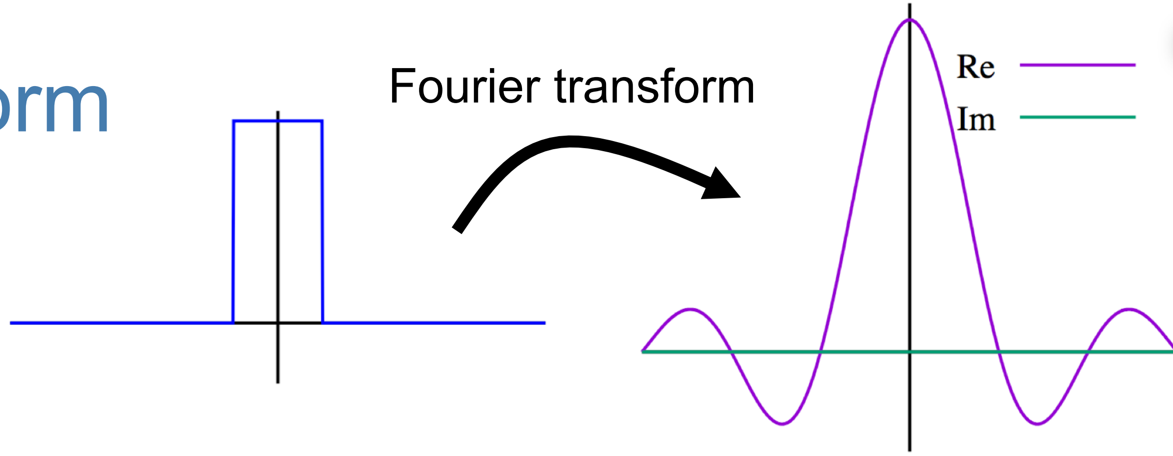
Example: Communication Channel is an LTI Filter



Impulse response: $h(t) = \delta(t-\tau_1) + \delta(t-\tau_2) + \delta(t-\tau_3) + \delta(t-\tau_4)$

Only time-invariant for a limited time period (coherence time)

Fourier Transform



Fourier transform: $X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

Exists if $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

Inverse transform: $\mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$

Common terminology

Amplitude spectrum: $|X(f)|$

Phase spectrum: $\arg\{X(f)\}$

Fourier Transform – Examples

Cosine:

$$x(t) = \cos(2\pi f_0 t)$$

$$X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Sine:

$$x(t) = \sin(2\pi f_0 t)$$

$$X(f) = \frac{1}{j2}\delta(f - f_0) - \frac{1}{j2}\delta(f + f_0)$$

Rectangle pulse:

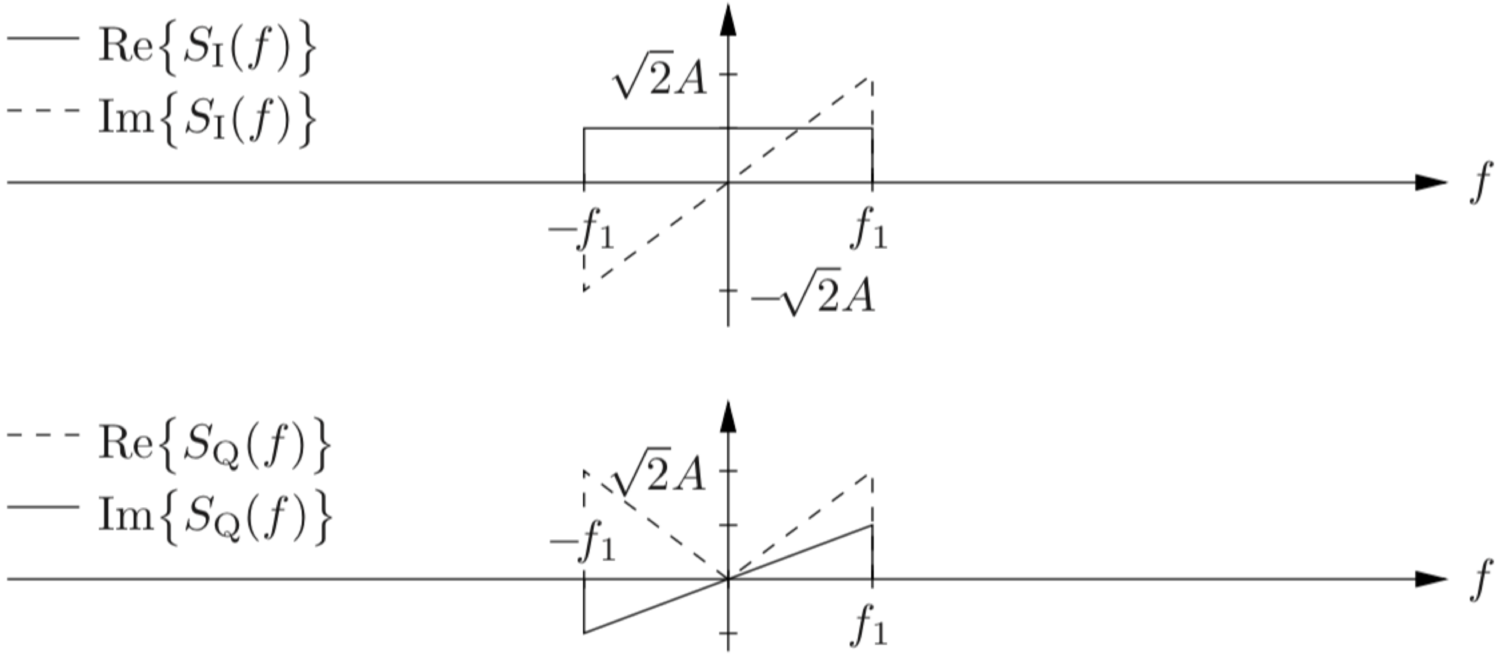
$$x(t) = u(t + 1) - u(t - 1)$$

$$X(f) = \mathrm{sinc}(f) = \frac{\sin(\pi f)}{\pi f}$$

Example: Two Baseband Signals

Consider two real-valued baseband signals $s_I(t), s_Q(t)$

- Bandwidth f_1
- Fourier transforms satisfy $S_I(f) = S_I^*(-f), S_Q(f) = S_Q^*(-f)$

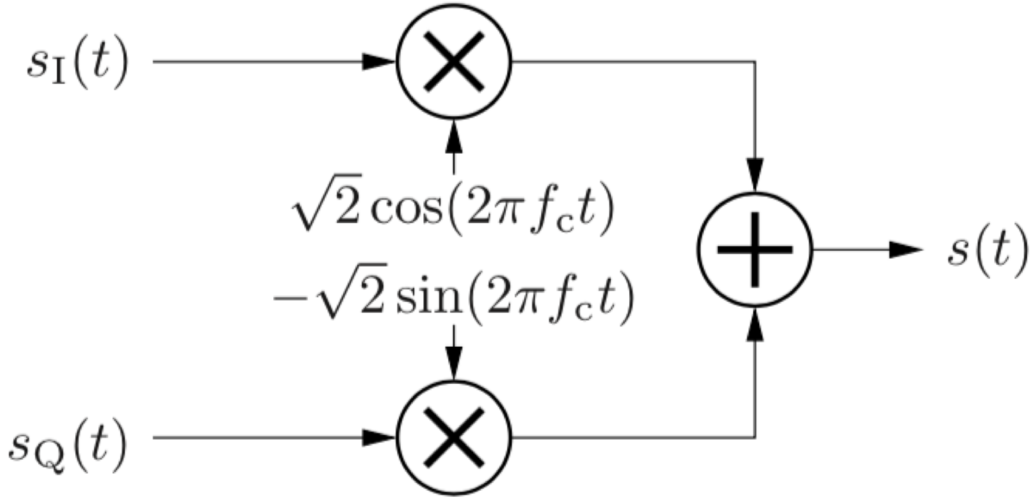


Modulation from Baseband to Passband

We need to communicate around the frequency f_c

- Create a passband signal from $s_I(t), s_Q(t)$:

$$s(t) = s_I(t)\sqrt{2}\cos(2\pi f_c t) - s_Q(t)\sqrt{2}\sin(2\pi f_c t)$$



Recall: Fourier Transform

Recall:

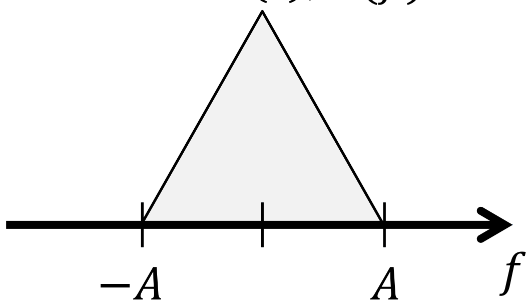
$$\mathcal{F}\{x(t)\} = X(f)$$

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

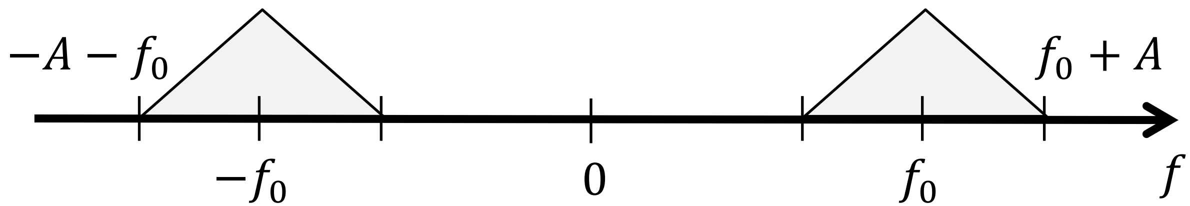
Consequence:

$$\mathcal{F}\{x(t)\cos(2\pi f_0 t)\} = \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$$

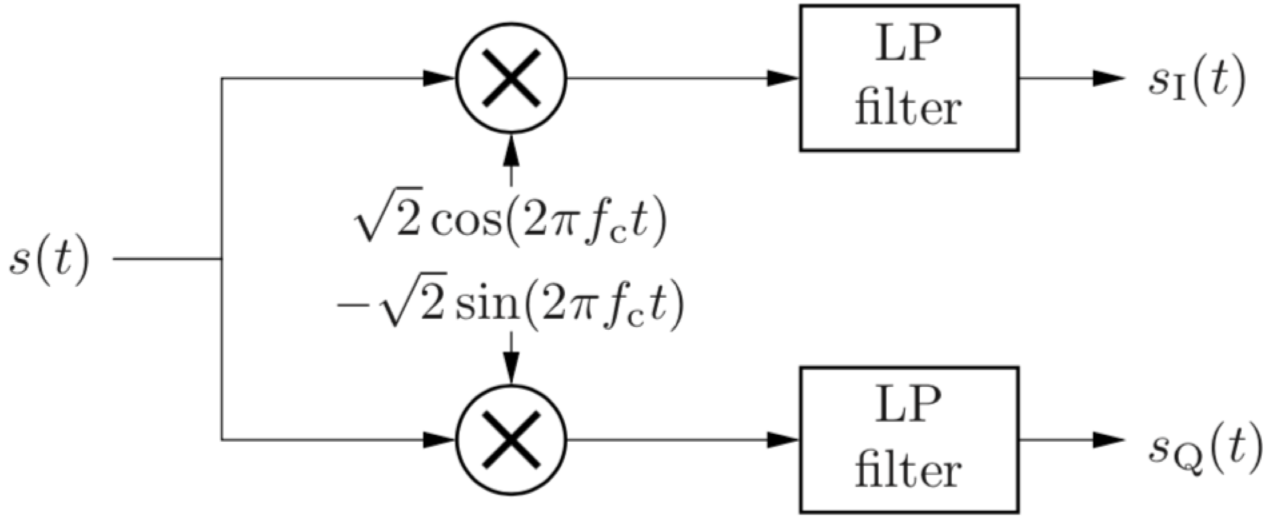
Baseband: $x(t), X(f)$



Passband: $x(t)\cos(2\pi f_0 t)$, $\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$



Demodulation from Passband to Baseband



Upper part:

$$
\begin{aligned}
&\sqrt{2}\cos(2\pi f_c t)\, s(t) \\
&= s_I(t) 2\cos^2(2\pi f_c t) - s_Q(t) 2\sin(2\pi f_c t)\cos(2\pi f_c t) \\
&= s_I(t) + \underbrace{s_I(t)\cos(4\pi f_c t) - s_Q(t)\sin(4\pi f_c t)}
\end{aligned}
$$

Centered around frequency $2f_c$; easy to cancel with low-pass filter

Complex Baseband Representation

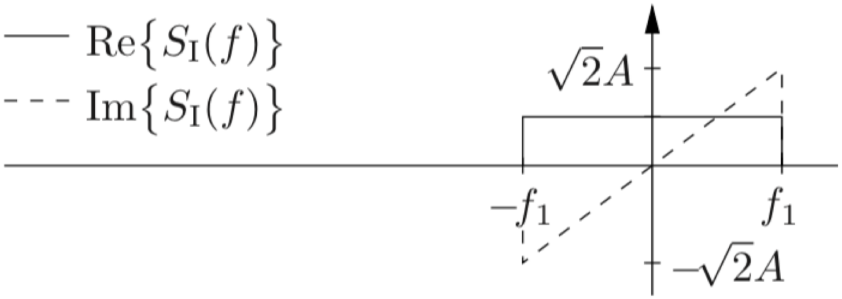
- Define a complex baseband signal instead:

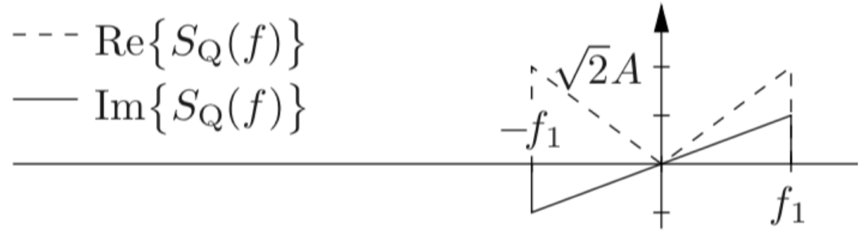
  $\tilde{s}(t) = s_I(t) + js_Q(t)$ with Fourier transform $\tilde{S}(f) = S_I(f) + jS_Q(f)$

- We can obtain passband signal as

$$s(t) = Re\left\{\sqrt{2}\tilde{s}(t)e^{j2\pi f_c t}\right\} = \frac{1}{\sqrt{2}}\left(\tilde{s}(t)e^{j2\pi f_c t} + \tilde{s}^*(t)e^{-j2\pi f_c t}\right)$$
$$= s_I(t)\sqrt{2}\cos(2\pi f_c t) - s_Q(t)\sqrt{2}\sin(2\pi f_c t)$$

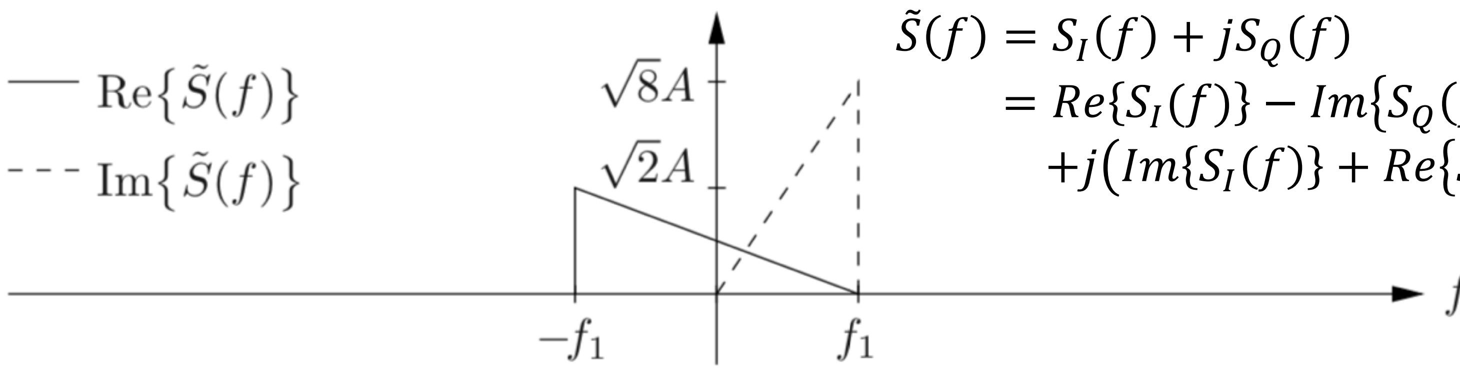
  - Hence: $S(f) = \frac{1}{\sqrt{2}}\left(\tilde{S}(f - f_c) + \tilde{S}^*(-f + f_c)\right)$





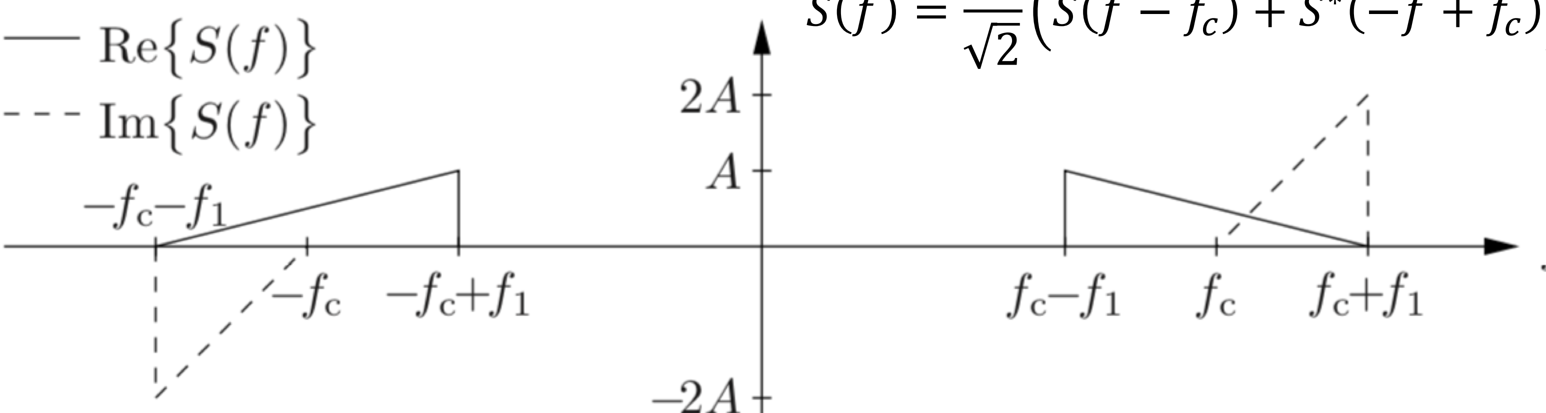
Spectrum of complex baseband signal $\tilde{s}(t)$

$$
\begin{aligned}
\tilde{S}(f) &= S_I(f) + jS_Q(f) \\
&= Re\{S_I(f)\} - Im\{S_Q(f)\} \\
&\quad +j\big(Im\{S_I(f)\} + Re\{S_Q(f)\}\big)
\end{aligned}
$$



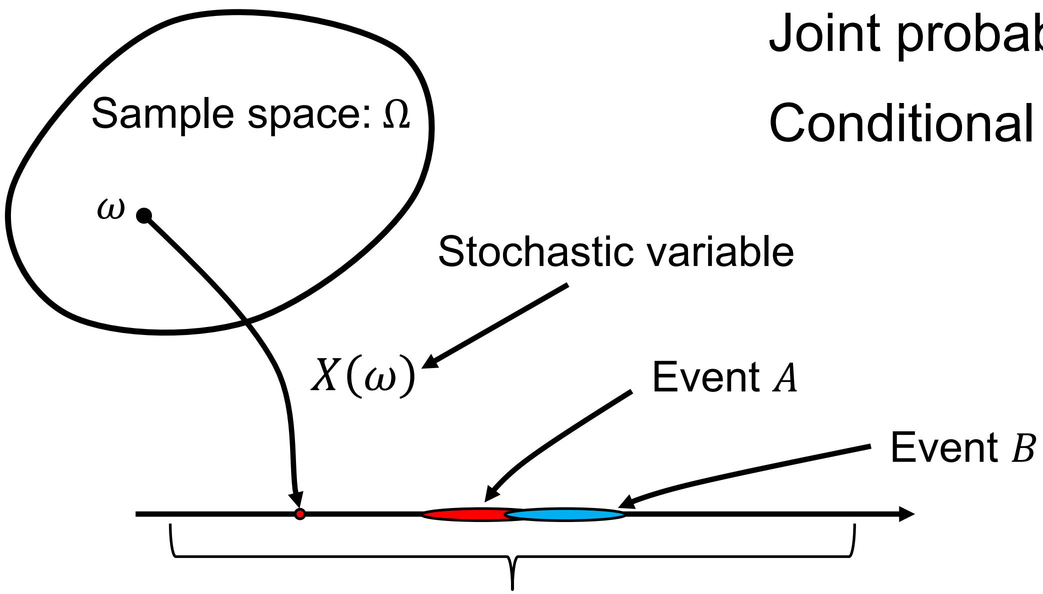
Spectrum of passband signal $s(t)$

$$
S(f) = \frac{1}{\sqrt{2}}\Big(\tilde{S}(f - f_c) + \tilde{S}^*(-f + f_c)\Big)
$$



Probability, Stochastic Variable, and Events

Total probability:	$\Pr\{\Omega_X\} = 1$
Probability of event A:	$\Pr\{A\} \in [0,1]$
Joint probability:	$\Pr\{A, B\}$
Conditional probability:	$\Pr\{A\|B\} = \frac{\Pr\{A,B\}}{\Pr\{B\}}$



Measureable sample space:
$\Omega_X = \{X(\omega)\colon \text{ for some } \omega \in \Omega\}$

Probability Theory

Stochastic variable X, taking realizations x

Probability distribution function:

$$F_X(x) = \Pr\{X \leq x\} \in [0,1]$$

Probability density function (PDF):

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Properties:

$F_X(x)$ is non-decreasing,

$F_X(x) \geq 0$ and $f_X(x) \geq 0$ for all x,

$\int_{-\infty}^{\infty} f_X(x)\, dx = 1,$

Expectation and Variance

Expectation (mean):

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx$$

For discrete variables:

$$E\{X\} = \sum_i x_i \Pr\{X = x_i\}$$

Quadratic mean (power):

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

Variance

$$\begin{aligned} \mathrm{Var}\{X\} &= E\{(X - E\{X\})^2\} \\ &= E\{X^2\} - (E\{X\})^2 \end{aligned}$$

Common notation:

$m_X = E\{X\}$, $m_Y = E\{Y\}$

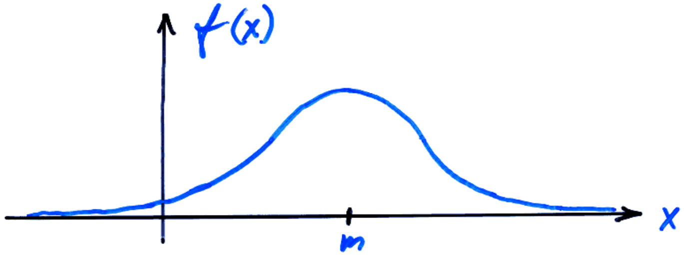
$\sigma_X^2 = \mathrm{Var}\{X\}$

σ_X is called the standard deviation

Gaussian/Normal Distribution, $N(m, \sigma^2)$

Probability density

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$



Mean: m

Variance: σ^2

Probability distribution $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$ is complicated

Q-function

$$Q(x) = 1 - F_X(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \quad \text{for } N(0,1)$$

Can be computed from table

The Q-function

Table of $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ for $0.00 \le x \le 5.99$.

x	0	1	2	3	4	5	6	7	8	9	exp
0.0	5.0000	4.9601	4.9202	4.8803	4.8405	4.8006	4.7608	4.7210	4.6812	4.6414	
0.1	4.6017	4.5620	4.5224	4.4828	4.4433	4.4038	4.3644	4.3251	4.2858	4.2465	
0.2	4.2074	4.1683	4.1294	4.0905	4.0517	4.0129	3.9743	3.9358	3.8974	3.8591	
0.3	3.8209	3.7828	3.7448	3.7070	3.6693	3.6317	3.5942	3.5569	3.5197	3.4827	
0.4	3.4458	3.4090	3.3724	3.3360	3.2997	3.2636	3.2276	3.1918	3.1561	3.1207	
0.5	3.0854	3.0503	3.0153	2.9806	2.9460	2.9116	2.8774	2.8434	2.8096	2.7760	
0.6	2.7425	2.7093	2.6763	2.6435	2.6109	2.5785	2.5463	2.5143	2.4825	2.4510	−1
0.7	2.4196	2.3885	2.3576	2.3270	2.2965	2.2663	2.2363	2.2065	2.1770	2.1476	
0.8	2.1186	2.0897	2.0611	2.0327	2.0045	1.9766	1.9489	1.9215	1.8943	1.8673	
0.9	1.8406	1.8141	1.7879	1.7619	1.7361	1.7106	1.6853	1.6602	1.6354	1.6109	
1.0	1.5866	1.5625	1.5386	1.5151	1.4917	1.4686	1.4457	1.4231	1.4007	1.3786	
1.1	1.3567	1.3350	1.3136	1.2924	1.2714	1.2507	1.2302	1.2100	1.1900	1.1702	
1.2	1.1507	1.1314	1.1123	1.0935	1.0749	1.0565	1.0383	1.0204	1.0027	9.8525	
1.3	9.6800	9.5098	9.3418	9.1759	9.0123	8.8508	8.6915	8.5343	8.3793	8.2264	
1.4	8.0757	7.9270	7.7804	7.6359	7.4934	7.3529	7.2145	7.0781	6.9437	6.8112	
1.5	6.6807	6.5522	6.4255	6.3008	6.1780	6.0571	5.9380	5.8208	5.7053	5.5917	
1.6	5.4799	5.3699	5.2616	5.1551	5.0503	4.9471	4.8457	4.7460	4.6479	4.5514	
1.7	4.4565	4.3633	4.2716	4.1815	4.0930	4.0059	3.9204	3.8364	3.7538	3.6727	−2
1.8	3.5930	3.5148	3.4380	3.3625	3.2884	3.2157	3.1443	3.0742	3.0054	2.9379	
1.9	2.8717	2.8067	2.7429	2.6803	2.6190	2.5588	2.4998	2.4419	2.3852	2.3295	
2.0	2.2750	2.2216	2.1692	2.1178	2.0675	2.0182	1.9699	1.9226	1.8763	1.8309	
2.1	1.7864	1.7429	1.7003	1.6586	1.6177	1.5778	1.5386	1.5003	1.4629	1.4262	
2.2	1.3903	1.3553	1.3209	1.2874	1.2545	1.2224	1.1911	1.1604	1.1304	1.1011	
2.3	1.0724	1.0444	1.0170	9.9031	9.6419	9.3867	9.1375	8.8940	8.6563	8.4242	
2.4	8.1975	7.9763	7.7603	7.5494	7.3436	7.1428	6.9469	6.7557	6.5691	6.3872	
2.5	6.2097	6.0366	5.8677	5.7031	5.5426	5.3861	5.2336	5.0849	4.9400	4.7988	
2.6	4.6612	4.5271	4.3965	4.2692	4.1453	4.0246	3.9070	3.7926	3.6811	3.5726	
2.7	3.4670	3.3642	3.2641	3.1667	3.0720	2.9798	2.8901	2.8028	2.7179	2.6354	−3
2.8	2.5551	2.4771	2.4012	2.3274	2.2557	2.1860	2.1182	2.0524	1.9884	1.9262	
2.9	1.8658	1.8071	1.7502	1.6948	1.6411	1.5889	1.5382	1.4890	1.4412	1.3949	
3.0	1.3499	1.3062	1.2639	1.2228	1.1829	1.1442	1.1067	1.0703	1.0350	1.0008	
3.1	9.6760	9.3544	9.0426	8.7403	8.4474	8.1635	7.8885	7.6219	7.3638	7.1136	
3.2	6.8714	6.6367	6.4095	6.1895	5.9765	5.7703	5.5706	5.3774	5.1904	5.0094	
3.3	4.8342	4.6648	4.5009	4.3423	4.1889	4.0406	3.8971	3.7584	3.6243	3.4946	
3.4	3.3693	3.2481	3.1311	3.0179	2.9086	2.8029	2.7009	2.6023	2.5071	2.4151	−4
3.5	2.3263	2.2405	2.1577	2.0778	2.0006	1.9262	1.8543	1.7849	1.7180	1.6534	
3.6	1.5911	1.5310	1.4730	1.4171	1.3632	1.3112	1.2611	1.2128	1.1662	1.1213	
3.7	1.0780	1.0363	9.9611	9.5740	9.2010	8.8417	8.4957	8.1624	7.8414	7.5324	
3.8	7.2348	6.9483	6.6726	6.4072	6.1517	5.9059	5.6694	5.4418	5.2228	5.0122	
3.9	4.8096	4.6148	4.4274	4.2473	4.0741	3.9076	3.7475	3.5936	3.4458	3.3037	−5
4.0	3.1671	3.0359	2.9099	2.7888	2.6726	2.5609	2.4536	2.3507	2.2518	2.1569	
4.1	2.0658	1.9783	1.8944	1.8138	1.7365	1.6624	1.5912	1.5230	1.4575	1.3948	
4.2	1.3346	1.2769	1.2215	1.1685	1.1176	1.0689	1.0221	9.7736	9.3447	8.9337	
4.3	8.5399	8.1627	7.8015	7.4555	7.1241	6.8069	6.5031	6.2123	5.9340	5.6675	
4.4	5.4125	5.1685	4.9350	4.7117	4.4979	4.2935	4.0980	3.9110	3.7322	3.5612	−6
4.5	3.3977	3.2414	3.0920	2.9492	2.8127	2.6823	2.5577	2.4386	2.3249	2.2162	
4.6	2.1125	2.0133	1.9187	1.8283	1.7420	1.6597	1.5810	1.5060	1.4344	1.3660	
4.7	1.3008	1.2386	1.1792	1.1226	1.0686	1.0171	9.6796	9.2113	8.7648	8.3391	
4.8	7.9333	7.5465	7.1779	6.8267	6.4920	6.1731	5.8693	5.5799	5.3043	5.0418	
4.9	4.7918	4.5538	4.3272	4.1115	3.9061	3.7107	3.5247	3.3476	3.1792	3.0190	−7
5.0	2.8665	2.7215	2.5836	2.4524	2.3277	2.2091	2.0963	1.9891	1.8872	1.7903	
5.1	1.6983	1.6108	1.5277	1.4487	1.3737	1.3024	1.2347	1.1705	1.1094	1.0515	
5.2	9.9644	9.4420	8.9462	8.4755	8.0288	7.6050	7.2028	6.8212	6.4592	6.1158	
5.3	5.7901	5.4813	5.1884	4.9106	4.6473	4.3977	4.1611	3.9368	3.7243	3.5229	
5.4	3.3320	3.1512	2.9800	2.8177	2.6640	2.5185	2.3807	2.2502	2.1266	2.0097	−8
5.5	1.8990	1.7942	1.6950	1.6012	1.5124	1.4283	1.3489	1.2737	1.2026	1.1353	
5.6	1.0718	1.0116	9.5479	9.0105	8.5025	8.0224	7.5686	7.1399	6.7347	6.3520	
5.7	5.9904	5.6488	5.3262	5.0215	4.7338	4.4622	4.2057	3.9636	3.7350	3.5193	
5.8	3.3157	3.1236	2.9424	2.7714	2.6100	2.4579	2.3143	2.1790	2.0513	1.9310	−9
5.9	1.8175	1.7105	1.6097	1.5147	1.4251	1.3407	1.2612	1.1863	1.1157	1.0492	

The exponent is found in the column labeled "exp". Ex. $Q(3.71) = 1.0363 \cdot 10^{-4}$.

For $x > 0$, we have $(1 - x^{-2})\frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt < Q(x) < \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt$. For large x we have $Q(x) \approx \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}dt$.

Example of the Q Function

x	0	1	2	3	4	5	6	7	8	9	exp
0.0	5.0000	4.9601	4.9202	4.8803	4.8405	4.8006	4.7608	4.7210	4.6812	4.6414	
0.1	4.6017	4.5620	4.5224	4.4828	4.4433	4.4038	4.3644	4.3251	4.2858	4.2465	
0.2	4.2074	4.1683	4.1294	4.0905	4.0517	4.0129	3.9743	3.9358	3.8974	3.8591	
0.3	3.8209	3.7828	3.7448	3.7070	3.6693	3.6317	3.5942	3.5569	3.5197	3.4827	
0.4	3.4458	3.4090	3.3724	3.3360	3.2997	3.2636	3.2276	3.1918	3.1561	3.1207	
0.5	3.0854	3.0503	3.0153	2.9806	2.9460	2.9116	2.8774	2.8434	2.8096	2.7760	
0.6	2.7425	2.7093	2.6763	2.6435	2.6109	2.5785	2.5463	2.5143	2.4825	2.4510	-1
0.7	2.4196	2.3885	2.3576	2.3270	2.2965	2.2663	2.2363	2.2065	2.1770	2.1476	
0.8	2.1186	2.0897	2.0611	2.0327	2.0045	1.9766	1.9489	1.9215	1.8943	1.8673	
0.9	1.8406	1.8141	1.7879	1.7619	1.7361	1.7106	1.6853	1.6602	1.6354	1.6109	
1.0	1.5866	1.5625	1.5386	1.5151	1.4917	1.4686	1.4457	1.4231	1.4007	1.3786	
1.1	1.3567	1.3350	1.3136	1.2924	1.2714	1.2507	1.2302	1.2100	1.1900	1.1702	
1.2	1.1507	1.1314	1.1123	1.0935	1.0749	1.0565	1.0383	1.0204	1.0027	9.8525	
1.3	9.6800	9.5098	9.3418	9.1759	9.0123	8.8508	8.6915	8.5343	8.3793	8.2264	
1.4	8.0757	7.9270	7.7804	7.6359	7.4934	7.3529	7.2145	7.0781	6.9437	6.8112	
1.5	6.6807	6.5522	6.4255	6.3008	6.1780	6.0571	5.9380	5.8208	5.7053	5.5917	
1.6	5.4799	5.3699	5.2616	5.1551	5.0503	4.9471	4.8457	4.7460	4.6479	4.5514	
1.7	4.4565	4.3633	4.2716	4.1815	4.0930	4.0059	3.9204	3.8364	3.7538	3.6727	-2
1.8	3.5930	3.5148	3.4380	3.3625	3.2884	3.2157	3.1443	3.0742	3.0054	2.9379	
1.9	2.8717	2.8067	2.7429	2.6803	2.6190	2.5588	2.4998	2.4419	2.3852	2.3295	
2.0	2.2750	2.2216	2.1692	2.1178	2.0675	2.0182	1.9699	1.9226	1.8763	1.8309	
2.1	1.7864	1.7429	1.7003	1.6586	1.6177	1.5778	1.5386	1.5003	1.4629	1.4262	
2.2	1.3903	1.3553	1.3209	1.2874	1.2545	1.2224	1.1911	1.1604	1.1304	1.1011	
2.3	1.0724	1.0444	1.0170	9.9031	9.6419	9.3867	9.1375	8.8940	8.6563	8.4242	

$$Q(1.96) \approx 2.4998 \cdot 10^{-2}$$

Bayes' Theorem

Thomas Bayes

Joint and conditional probability:

$$\Pr\{X = x, Y = y\} = \Pr\{X = x | Y = y\} \Pr\{Y = y\}$$
$$= \Pr\{Y = y | X = x\} \Pr\{X = x\}$$

$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y) = f_{Y|X}(y|x) f_X(x)$$

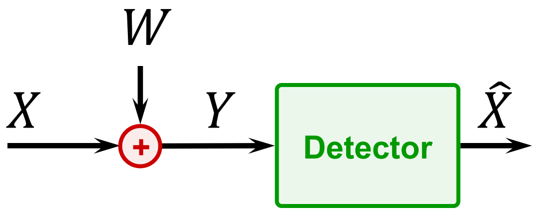
Bayes' theorem (discrete): $\Pr\{X = x | Y = y\} = \frac{\Pr\{Y=y|X=x\}}{\Pr\{Y=y\}} \Pr\{X = x\}$

(continuous): $f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)}{f_Y(y)} f_X(x)$

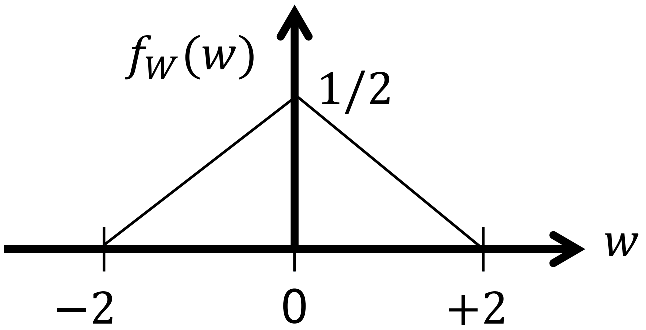
(X discrete, Y cont.): $\Pr\{X = x | Y = y\} = \frac{f_{Y|X}(y|x)}{f_Y(y)} \Pr\{X = x\}$

Image from: https://en.wikipedia.org/wiki/Thomas_Bayes

Example: Additive Noise Channel



- Input: $X \in \{-1, +1\}$ with $\Pr\{X = +1\} = \Pr\{X = -1\} = 1/2$



- Output: $Y = X + W$
  - W is a continuous stochastic variable

How should we “best” detect X from Y?

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