

# TSDT14 Signal Theory

## Lecture 11

### Multi-Dimensional Processes – Primarily 2-D

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## Example: 2-D Filtering

Mr. ?  
Angry



? Mrs.  
Calm

Source: Aude Olivia & Philippe G. Schyns, MIT.  
<http://cvcl.mit.edu/hybridimage/>

## Example: 2-D Filtering

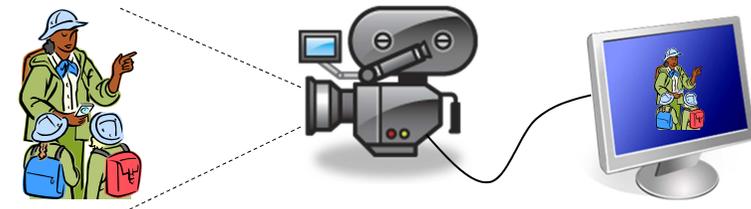
Mr.  
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## Multi-Dimensional Signals & Systems

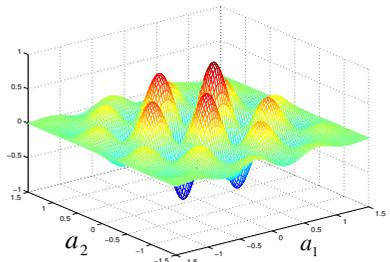


## Two-Dimensional Signals

A function of two variables

$x(a_1, a_2)$  (space-continuous)

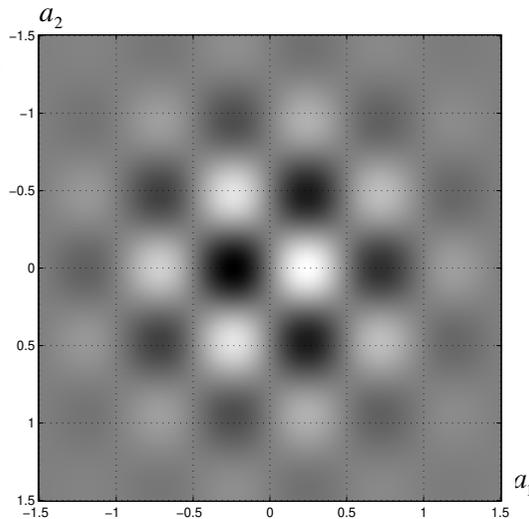
$x[n_1, n_2]$  (space-discrete)



Separable signals:

$$x(a_1, a_2) = x_1(a_1) \cdot x_2(a_2)$$

$$x[n_1, n_2] = x_1[n_1] \cdot x_2[n_2]$$



## Classification of Systems

Linearity:

Input  $x_1(a_1, a_2)$  gives output  $y_1(a_1, a_2)$ .

Input  $x_2(a_1, a_2)$  gives output  $y_2(a_1, a_2)$ .

Then input  $a \cdot x_1(a_1, a_2) + b \cdot x_2(a_1, a_2)$   
gives output  $a \cdot y_1(a_1, a_2) + b \cdot y_2(a_1, a_2)$ .

Space-invariance:

Input  $x(a_1, a_2)$  gives output  $y(a_1, a_2)$ .

Then input  $x(a_1 - b_1, a_2 - b_2)$   
gives output  $y(a_1 - b_1, a_2 - b_2)$ .

LSI:

Both linear and space-invariant.

Similarly for space-discrete systems

## Two-Dimensional Stochastic Processes

Mean:  $m_X(a_1, a_2) = E\{X(a_1, a_2)\}$

Auto-correlation:  $r_X(a_1, a_2; a_1 + b_1, a_2 + b_2) = E\{X(a_1, a_2)X(a_1 + b_1, a_2 + b_2)\}$

Wide-sense stationarity:

Both  $m_X(a_1, a_2)$  and  $r_X(a_1, a_2; a_1 + b_1, a_2 + b_2)$  independent of  $(a_1, a_2)$ .

Simplified notation:  $m_X$  and  $r_X(b_1, b_2)$

## Two-Dimensional Convolution

For LSI systems, the output is given by a two-dimensional convolution of the input and the impulse response of the filter.

Definition:

$$(x \otimes h)(a_1, a_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(b_1, b_2) h(a_1 - b_1, a_2 - b_2) db_1 db_2$$

Separable signals:

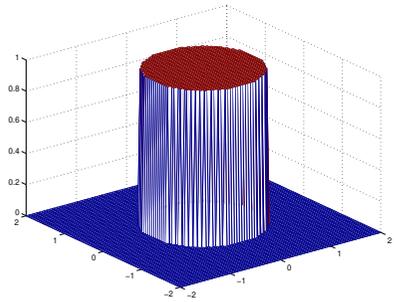
$$\begin{aligned} (x \otimes h)(a_1, a_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(b_1) x_2(b_2) h_1(a_1 - b_1) h_2(a_2 - b_2) db_1 db_2 \\ &= \int_{-\infty}^{\infty} x_1(b_1) h_1(a_1 - b_1) db_1 \int_{-\infty}^{\infty} x_2(b_2) h_2(a_2 - b_2) db_2 \\ &= (x_1 * h_1)(a_1) \cdot (x_2 * h_2)(a_2) \end{aligned}$$

Space-discrete:

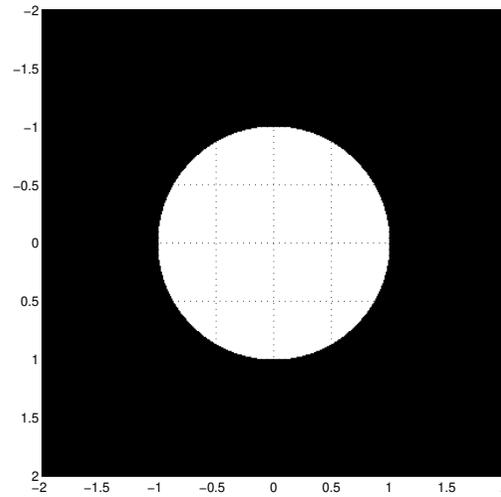
$$(x \otimes h)[n_1, n_2] = \sum_{k_1} \sum_{k_2} x[k_1, k_2] \cdot h[n_1 - k_1, n_2 - k_2]$$

## Non-Separable Convolution 1(2)

$$x(a_1, a_2) = h(a_1, a_2) = \begin{cases} 1, & a_1^2 + a_2^2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

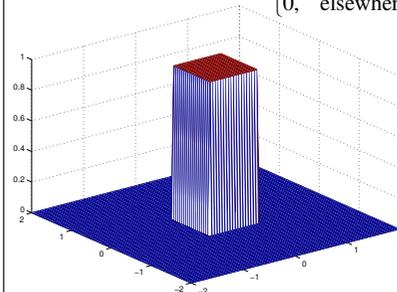


Non-separable signals

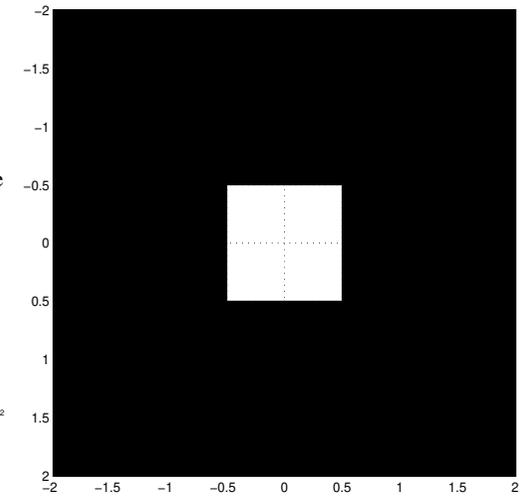


## Separable Convolution 1(3)

$$\begin{aligned} x(a_1, a_2) &= h(a_1, a_2) = \\ &= \begin{cases} 1, & |a_1| < 1/2 \text{ and } |a_2| < 1/2 \\ 0, & \text{elsewhere} \end{cases} \\ &= z(a_1) \cdot z(a_2) \\ \text{with } z(a) &= \text{rect}(a) = \begin{cases} 1, & |a| < 1/2 \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$

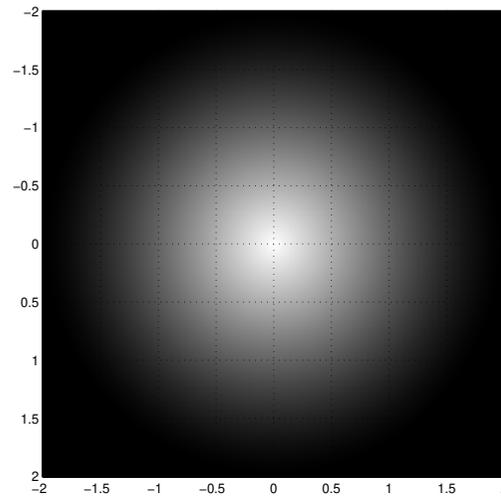
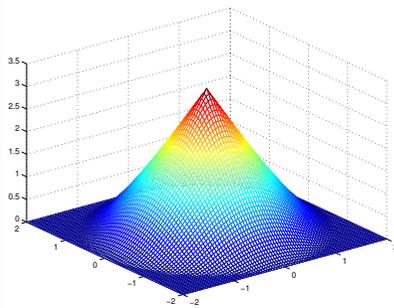


Separable signals



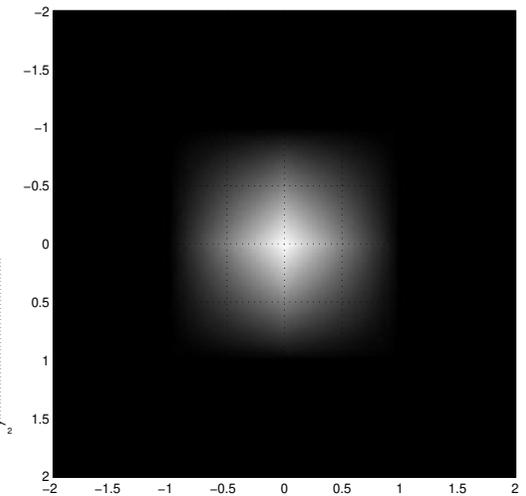
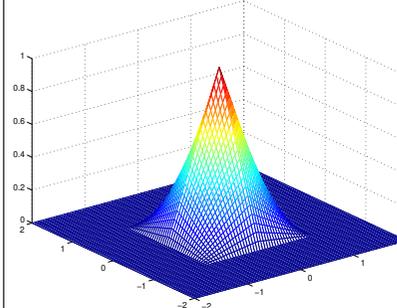
## Non-Separable Convolution 2(2)

$$\begin{aligned} y(a_1, a_2) &= (x \otimes h)(a_1, a_2) \\ &= \begin{cases} 2 \arccos(a/2) - a \sqrt{1 - (a/2)^2}, & a^2 = a_1^2 + a_2^2 < 4 \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$



## Separable Convolution 2(3)

$$\begin{aligned} y(a_1, a_2) &= (x \otimes h)(a_1, a_2) \\ &= \max\{0, 1 - |a_1|\} \cdot \max\{0, 1 - |a_2|\} \\ &= \text{triangle}(a_1) \cdot \text{triangle}(a_2) \end{aligned}$$

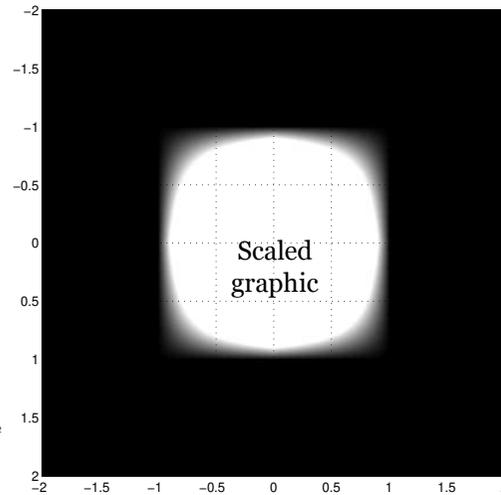
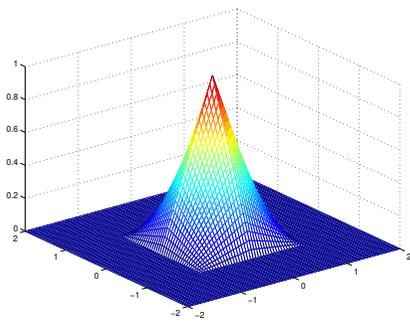


## Separable Convolution 3(3)

$$y(a_1, a_2) = (x \otimes h)(a_1, a_2)$$

$$= \max\{0, 1 - |a_1|\} \cdot \max\{0, 1 - |a_2|\}$$

$$= \text{triangle}(a_1) \cdot \text{triangle}(a_2)$$



## 2-D Space-Discrete Fourier Transform

Definition:  $X[\theta_1, \theta_2] = \mathcal{F}\{x[n_1, n_2]\} = \sum_{n_1} \sum_{n_2} x[n_1, n_2] e^{-j2\pi(\theta_1 n_1 + \theta_2 n_2)}$

Inverse:  $x[a_1, a_2] = \mathcal{F}^{-1}\{X[\theta_1, \theta_2]\} = \int_0^1 \int_0^1 X[\theta_1, \theta_2] e^{j2\pi(\theta_1 a_1 + \theta_2 a_2)} d\theta_1 d\theta_2$

Properties:  $X[\theta_1, \theta_2] = X[\theta_1 + m_1, \theta_2 + m_2]$  for integers  $m_1$  and  $m_2$ .

$$\mathcal{F}\{x[n_1, n_2]\} = \mathcal{F}_1\{\mathcal{F}_2\{x[n_1, n_2]\}\} = \mathcal{F}_2\{\mathcal{F}_1\{x[n_1, n_2]\}\}$$

$$\mathcal{F}\{x_1[n_1] \cdot x_2[n_2]\} = \mathcal{F}_1\{x_1[n_1]\} \cdot \mathcal{F}_2\{x_2[n_2]\}$$

$$\mathcal{F}\{(x \otimes h)[n_1, n_2]\} = X[\theta_1, \theta_2] \cdot H[\theta_1, \theta_2]$$

$$\mathcal{F}\{x[n_1, n_2] \cdot h[n_1, n_2]\} = (X \otimes H)[\theta_1, \theta_2] \quad (\text{periodic conv})$$

## 2-D Space-Continuous Fourier Transform

Definition:  $X(f_1, f_2) = \mathcal{F}\{x(a_1, a_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(a_1, a_2) e^{-j2\pi(f_1 a_1 + f_2 a_2)} da_1 da_2$

Inverse:  $x(a_1, a_2) = \mathcal{F}^{-1}\{X(f_1, f_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f_1, f_2) e^{j2\pi(f_1 a_1 + f_2 a_2)} df_1 df_2$

Properties:  $\mathcal{F}\{x(a_1, a_2)\} = \mathcal{F}_1\{\mathcal{F}_2\{x(a_1, a_2)\}\} = \mathcal{F}_2\{\mathcal{F}_1\{x(a_1, a_2)\}\}$

$$\mathcal{F}\{x_1(a_1) \cdot x_2(a_2)\} = \mathcal{F}_1\{x_1(a_1)\} \cdot \mathcal{F}_2\{x_2(a_2)\}$$

$$\mathcal{F}\{(x \otimes h)(a_1, a_2)\} = X(f_1, f_2) \cdot H(f_1, f_2)$$

$$\mathcal{F}\{x(a_1, a_2) \cdot h(a_1, a_2)\} = (X \otimes H)(f_1, f_2)$$

## Power and Power Spectral Density

PSD:  $R_X(f_1, f_2) = \mathcal{F}\{r_X(b_1, b_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_X(b_1, b_2) e^{-j2\pi(f_1 b_1 + f_2 b_2)} db_1 db_2$

Power:  $P_X = E\{X^2(a_1, a_2)\} = r_X(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(f_1, f_2) df_1 df_2$

Output of LSI-system if input is WSS:

Output:  $Y(a_1, a_2) = (X \otimes h)(a_1, a_2)$

Mean:  $m_Y = m_X \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(a_1, a_2) da_1 da_2 = m_X \cdot H(0, 0)$

ACF:  $r_Y(b_1, b_2) = (h \otimes \tilde{h} \otimes r_X)(b_1, b_2)$  with  $\tilde{h}(a_1, a_2) = h(-a_1, -a_2)$

PSD:  $R_Y(f_1, f_2) = |H(f_1, f_2)|^2 R_X(f_1, f_2)$

## Two-Dimensional Sampling

Deterministic (for rectangular grid):

Sampling:  $y[n_1, n_2] = x(n_1 A_1, n_2 A_2)$

Sampling periods:  $A_1$  and  $A_2$

Spectrum: 
$$Y[\theta_1, \theta_2] = \frac{1}{A_1 A_2} \sum_{k_1} \sum_{k_2} X\left(\frac{\theta_1 - k_1}{A_1}, \frac{\theta_2 - k_2}{A_2}\right)$$

Probabilistic (for rectangular grid):

Sampling:  $Y[n_1, n_2] = X(n_1 A_1, n_2 A_2)$

PSD: 
$$R_Y[\theta_1, \theta_2] = \frac{1}{A_1 A_2} \sum_{k_1} \sum_{k_2} R_X\left(\frac{\theta_1 - k_1}{A_1}, \frac{\theta_2 - k_2}{A_2}\right)$$

## Two-Dimensional PAM

Deterministic (for rectangular grid):

PAM: 
$$z(a_1, a_2) = \sum_{n_1} \sum_{n_2} y[n_1, n_2] p(a_1 - n_1 A_1, a_2 - n_2 A_2)$$

Spectrum:  $Z(f_1, f_2) = P(f_1, f_2) \cdot Y[f_1 A_1, f_2 A_2]$

Probabilistic (for rectangular grid):

PAM: 
$$Z(a_1, a_2) = \sum_{n_1} \sum_{n_2} Y[n_1, n_2] p(a_1 - n_1 A_1 - \Psi_1, a_2 - n_2 A_2 - \Psi_2)$$

$\Psi_1$  uniform on  $[0, A_1)$  and  $\Psi_2$  uniform on  $[0, A_2)$   
both independent of  $Y[n_1, n_2]$  and of each other.

PSD: 
$$R_Z(f_1, f_2) = \frac{1}{A_1 A_2} |P(f_1, f_2)|^2 R_Y[f_1 A_1, f_2 A_2]$$

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